# On supply-function equilibria in radial transmission networks

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### Abstract

Transmission constraints limit competition and arbitrageurs' possibilities of exploiting price differences between commodities in neighbouring markets. We analyze radial transmission-constrained networks with local demand shocks, where spatially distributed oligopoly producers compete with supply functions, as in wholesale electricity markets. We prove existence and uniqueness of supply-function equilibrium in two-node networks, and we are able to explicitly solve for symmetric supply-function equilibria in two-node and star networks.

**Key words**: OR in energy, Spatial competition, Supply-function equilibrium, Transmission network, Wholesale electricity markets

**JEL Classification** C72, D43, D44, L91

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# 1    Introduction

In this paper, we analyse how transmission constraints influence competition in commodity markets with spatially distributed oligopoly producers. We consider a homogeneous commodity that is produced and consumed in local oligopoly markets connected by a network of transport links. Our analysis is mainly motivated by the design and operation of wholesale electricity markets. Transportation through a network link is costless up to the link's transport capacity. Demand is inelastic up to a reservation price and production costs are common knowledge. We consider a simultaneous-move game, where each strategic producer first commits to a supply function, as in wholesale electricity markets, and then a local exogenous additive demand shock is realized in each local market. After demand shocks have been realized, a price-taking transport company (such as a regulated network operator) buys the commodity at the cheap end of a transport link and sells it at the more expensive end, until the transport capacity is exhausted or until market prices are equal at both ends of the link. We solve for a Nash equilibrium (NE) of supply functions, also called a supply-function equilibrium (SFE).

The SFE for a single market with marginal (uniform) pricing was originally developed by Klemperer and Meyer (1989). The setting of the SFE is particularly well suited for markets where producers submit supply functions to a uniform-price auction before demand has been realized, as in wholesale electricity markets (Green and Newbery, 1992; Bolle 1992; Anderson and Philpott, 2002b; Sioshansi and Oren, 2007; Wolak, 2007; Hortaçsu and Puller, 2008; Holmberg and Newbery, 2010). In Klemperer and Meyer (1989), a producer can, in equilibrium, predict the slope of its residual demand at every price, so equilibrium offers are ex-post optimal. In this case, the optimal output of a producer is proportional to its mark-up and the slope of its residual demand at every price.

In our setting, transmission constraints and multiple local demand shocks mean that a firm does not know the slope of its residual demand with certainty at a given price, so offers should be chosen to be ex-ante optimal. We prove that Klemperer and Meyer's (1989) condition can be generalized to such cases; the optimal output of a producer is proportional to its mark-up and the expected slope of its residual demand at every price. This relationship can be used to solve for equilibria in radial networks.

Related results have been derived by Wilson (2008). We contribute by deriving global second-order conditions for general networks and by establishing uniqueness and existence results for networks with two nodes. Moreover, we contribute by deriving explicit expressions for symmetric SFE when producers have identical costs, producers are symmetrically distributed in star or two-node networks and multi-dimensional demand shocks are uniform. Related is also Anderson et al. (2007), who analyse how the best response of a producer changes when its network becomes interconnected to a previously separate grid with price-taking competitors. Recent applications of supply-function equilibria in networks can be found in the papers by Ruddell et al (2016), where all producers are in one node, and Khazaei et al (2017), who consider linear supply functions and multisettlements.

Verifying global optimality conditions is important for oligopoly markets with

transport constraints, because previous research has shown that such conditions are often violated in such settings. The reason is that transport constraints can result in a producer's residual demand curve having discontinuous changes in its slope. These kinks are such that the slope becomes discontinuously less price sensitive when net imports to the producer's local market are congested. In the neighbourhood of such a kink the residual demand curve is sufficiently convex to yield a profitable deviation from a first-order solution in which imports to a local market are nearly congested (namely by withholding production in order to push the price above the next breakpoint in its residual demand curve). This type of deviation will often rule out pure-strategy Nash equilibria in networks with transport constraints, especially if there are no market uncertainties, so that each bidder can perfectly predict the location of the kinks/breakpoints of its residual demand curve in equilibrium.

In our study we use Anderson and Philpott's (2002a) market-distribution-function approach to verify that monotonic solutions to our first-order conditions are supply-function equilibria (SFE) when the probability density of the demand shocks (shock density) is sufficiently evenly distributed, i.e. sufficiently close to a uniform multi-dimensional distribution. In this case the producers react to the expectation of the residual demand slope over different congestion conditions. The uncertainty has a smoothing effect, which reduces problems with local convexities in the residual demand curve, so that profitable deviations from first-order solutions can be precluded. But existence of SFE cannot be taken for granted. Profitable deviations from the first-order solution will for example exist for perfectly correlated demand shocks or for steep slopes and discontinuities in the probability density of the demand shocks.

There are several papers that have addressed different aspects of the existence problem for NE in power networks. Borenstein et al. (2000) for example rule out Cournot NE when the transport capacity between two symmetric markets is sufficiently small and demand is certain. Downward et al. (2010) analyse similar problems in general networks with transport constraints. Related are also Adler et al. (2008) and Hu and Ralph (2007) who show that existence of pure-strategy Cournot NE depends on the assumptions made about the rationality of the players. Escobar and Jofre (2006,2008) establish that a mixed-strategy NE normally exists in constrained transmission networks. Hobbs et al. (2004) bypasses the existence issue by using conjectural variations instead of a Nash equilibrium. Existence of equilibria is more straightforward in competitive networks with infinitesimally small producers (Cho, 2003; Escobar and Jofre, 2006, 2008; Holmberg and Lazarczyk, 2015).

It follows from Klemperer and Meyer (1989) and Genc and Reynolds (2011) that there will be multiple SFE in a single market when the demand shock is sufficiently bounded such that a producer is certain to sell a strictly positive output that is strictly lower than its production capacity. The reason is that a producer has a lot of freedom when choosing the shape of sections of a supply function that is never going to be price-setting. As illustrated by Klemperer and Meyer (1989), producers can use this freedom to support a wide range of equilibria. However,

as shown by Holmberg (2008) and Anderson (2013), a unique equilibrium will normally exist if demand shocks are such that any point of a producer's supply function would be marginal for some possible demand shock outcome. Our local demand shocks have this property, so our uniqueness results are consistent with the previous SFE literature for single markets.

The SFE model has mainly been used in studies of producers' strategic bidding in wholesale electricity markets. But there are some exceptions. Laussel (1992) and Pehlivan and Vuong (2013) have for example used the SFE to study competition between exporters in a global economy. In this context our model of radial networks with transport constraints would for example be useful when analysing the effect of trading quotas on the strategic interaction between exporters. Krishna (1989) has previously analysed such problems for the case with Bertrand competition between exporters. Related is also Malamud and Rostek's (2013) study of traders that compete with linear supply functions in a network of decentralized exchanges without transport constraints.

Normally a local market would represent the geographical location of a market place, and with transport we normally mean that the commodity is moved from one geographical location to another location. But local markets and transports could be interpreted in a more general sense. For example, a local market could represent a geographical location at a particular point in time. Thus storage can be represented by transport links that allow for transports of the commodity to the same place but at a later point in time, as in a time-expanded graph as described in Ford and Fulkerson (1962).

Our contributions in this paper can be summarized in the following way:

1. We derive conditions for computing the best response of a producer at a node of a radial transmission network;

2. We demonstrate existence and uniqueness of SFE in a two-node network;

3. We compute symmetric SFE in two-node and star networks, and show how these relate to a market integration function that can be computed from a model with price-taking agents;

4. We provide examples where SFE fail to exist.

The paper is laid out as follows. In Section 2 we derive necessary and sufficient conditions for optimality of a producer's supply function in a radial network. This is used to derive conditions for symmetric equilibrium. Section 3 provides three examples where we apply the techniques of Section 2 to compute supply function equilibria in radial networks. The first two examples study a symmetric two node network, and contrasts a demand shock distribution for which a SFE exists with one for which SFE do not exist. The third example constructs an equilibrium for a star network that is symmetric with respect to producers. The paper concludes in Section 4. All proofs are in the Appendices.

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# 2 Optimality and equilibrium conditions

In this paper we restrict attention to *radial networks*, which have a tree structure. Such graphs have no loops, so there is a unique chain of transport links between any two local markets. Radial networks for example include hub-and-spoke and line networks. Although most electric power networks contain loops, radial networks are often used as a first approximation. Some market operators in Europe use radial approximations when clearing the wholesale electricity market. Thus producers in such markets would consider this radial approximation when preparing their bids. Cho (2003) uses an extensive radial network to approximate the electric power grid in California. In the Nordic countries, Britain, New Zealand and Germany, the dominating transmission capacity constraints approximately separate the electricity market into a northern and southern market, giving a simple two-node radial representation.

## 2.1 Model set-up

A homogenous commodity is traded across a network of $M$ local markets (nodes). They are connected by $K$ directed transport links (arcs) in the following way: at most one arc connects any pair of nodes, but there is a path (series of arcs) that can be followed from any one node to any other (i.e. the network is connected). Radial networks have $K = M - 1$, so that there is a unique path between any two nodes. As is standard in graph theory, the topology of the network can be described by a *node-arc incidence* matrix $\mathbf{A}$ (Bazaraa et al., 2009). This matrix $\mathbf{A}$ has a row for every node and a column for every arc, and $mk-$th element $a_{mk}$ defined as follows:

$$a_{mk} = \begin{cases} -1, & \text{if arc } k \text{ is oriented away from node } m, \\ 1, & \text{if arc } k \text{ is oriented towards node } m, \\ 0, & \text{otherwise.} \end{cases}$$

(We note that some authors adopt a different convention in which $a_{mk} = 1$ if arc $k$ is oriented away from node $m$, but the above definition is more convenient for our purposes.)

The transported quantity in arc $k$ is represented by the variable $t_k$ which can be positive or negative, the latter indicating a flow in the opposite direction from the orientation of the arc. Thus the $m$th row of $\mathbf{At}$ represents the flow of the commodity into node $m$ (imports) from the rest of the network. Transportation is assumed to be lossless and costless, but each arc $k$ has a capacity $\bar{t}_k$, so the vector $\mathbf{t}$ of arc flows satisfies

$$-\bar{\mathbf{t}} \leq \mathbf{t} \leq \bar{\mathbf{t}}. \tag{1}$$

At each node $m$ there are $N_m$ producers who play a simultaneous-move, one-shot game. Nodes with $N_m = 0$ are referred to as *nonstrategic nodes*. The other nodes with $N_m \geq 1$ are referred to as *strategic nodes*. A particular case of interest is *networks that are symmetric with respect to producers*. This means that producers

have identical costs, identical production capacities and that a producer which exchanges location with any other producer sees the same game as before.

Each producer offers a strictly increasing differentiable supply function

$$Q_{mn}(p), \, n = 1, 2, \ldots, N_m,$$

that defines how much each firm is prepared to supply at price $p$. Each function $Q_{mn}(p)$ defines a strictly increasing inverse supply function $T_{mn}(q)$ by the price $p_m$ at which $Q_{mn}(p) = q$. The inverse supply function of a producer corresponds to a statement of its marginal cost, which may not be truthful. We denote the total nodal supply in each node by $S_m(p_m) = \sum_{n=1}^{N_m} Q_{mn}(p_m)$ and the vector with such components by $\mathbf{s}(\mathbf{p})$. We also introduce $S_{m,-n}(p_m) = \sum_{j=1, j \neq n}^{N_m} Q_{mj}(p_m)$, which excludes the supply of firm $n$ from the nodal supply in node $m$. Since $S_m(p_m)$ is strictly increasing we can define the inverse nodal supply function $T_m(q)$ by the price $p_m$ at which $S_m(p_m) = q$.

For simplicity we assume that each firm is only active in one node. Similar to Wilson (2008), forward contracts and similar contracts could be considered by assuming that the output, offered supply and production capacity are net of contracts.

Demand in each node $m$ is given by a random local shock $\varepsilon_m$ having a known probability distribution $\mathbb{P}$ with compact convex support $\mathcal{E}$ and joint density denoted $f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)$. This condition implies that $\mathbb{P}(E) = 0$ for any $E \subseteq \mathcal{E}$ with Lebesgue measure 0. The demand shocks are realized after firms have committed to their supply functions. Our model does not consider price-response on the demand side. However, similar to Wilson (2008), the model could be generalized by representing a strategic consumer with demand $D_{mn}(p)$ by a supply function $Q_{mn}(p) = -D_{mn}(p)$.

We assume that the commodity is traded at the local market price of each node. In electric power networks this is called nodal pricing or locational marginal pricing (LMP) (Chao and Peck, 1996; Hogan, 1992; Bohn et al., 1984). There is no storage in the nodes. Hence, for each realization $\boldsymbol{\varepsilon}$, the market clearing must result in network flows $\mathbf{t}$ and local market prices $\mathbf{p}$, such that net-imports are equal to net-consumption in each node (consumption net of production).

$$\mathbf{At} = \boldsymbol{\varepsilon} - \mathbf{s}(\mathbf{p}). \tag{2}$$

We assume that the network operator chooses the dispatch such that the total stated production costs are minimized.

We let $C_{mn}(q)$ be the production cost of firm $n$ in node $m$. It is differentiable, convex and increasing. The profit earned by producer $n$ when dispatched quantity $q$ at price $p_m$ is

$$\Pi_{mn}(q, p_m) = q p_m - C_{mn}(q).$$

When solving for an SFE, i.e. a Nash equilibrium of supply-function bids, we assume that each producer is risk-neutral and chooses its supply function in order to maximize its expected profit.

## 2.2 Market-clearing conditions

As shown by Chao and Peck (1996), market clearing can be achieved by solving an economic dispatch problem of the following form.

$$\text{DP: } \min \quad \sum_{m=1}^{M} \int_{0}^{q_m} T_m(x)\, dx$$
$$\text{s.t.} \quad \mathbf{At} = \boldsymbol{\varepsilon} - \mathbf{q}$$
$$-\bar{\mathbf{t}} \leq \mathbf{t} \leq \bar{\mathbf{t}}.$$

For any $\boldsymbol{\varepsilon} \in \mathcal{E}$, DP has a feasible solution defined by $\mathbf{t} = \mathbf{0}$, $\mathbf{q} = \boldsymbol{\varepsilon}$, and for fixed $\boldsymbol{\varepsilon}$ the feasible region is convex, and compact since $\mathbf{q} = \boldsymbol{\varepsilon} - \mathbf{At}$ lies in the continuous image of a compact set. Thus DP has an optimal solution, and since the objective function is strictly convex in $\mathbf{q}$ (because $T_m(q_m)$ is strictly increasing) the solution yields a unique dispatch $\mathbf{q}$. For radial networks we have that $\mathbf{t}$ is determined from the unique dispatch. Moreover since DP has linear constraints, we also have the existence of Lagrange multiplier vectors $\mathbf{p}$ for the flow balance constraints, $\boldsymbol{\rho}$ for flows in the positive direction, and $\boldsymbol{\sigma}$ for flows in the negative direction, yielding the following necessary and sufficient optimality conditions of problem DP:

$$\mathbf{A}^{\top}\mathbf{p} = \boldsymbol{\rho} - \boldsymbol{\sigma},$$
$$\rho_k \geq 0, \ t_k \leq \bar{t}_k, \ \rho_k(\bar{t}_k - t_k) = 0, \ k = 1 \ldots K,$$
$$\sigma_k \geq 0, \ t_k \geq -\bar{t}_k, \ \sigma_k(\bar{t}_k + t_k) = 0, \ k = 1 \ldots K, \quad (3)$$
$$\mathbf{At} + \mathbf{q} = \boldsymbol{\varepsilon}$$
$$\mathbf{q} = \mathbf{s}(\mathbf{p}).$$

It follows from Berge's Theorem of the maximum (Berge, 1963, p.116) that the optimal flows and optimal dispatched quantities are continuous with respect to the vector of demand shocks $\boldsymbol{\varepsilon}$. It follows from the market clearing conditions that this will also be the case for nodal prices and the Lagrange multipliers $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$.

According to the market clearing conditions, the network operator is equivalent to a price-taking transport company that buys the commodity at the cheap end of a transport link and sells it at the more expensive end, until the link's transport capacity is exhausted or until market prices are equal at both ends of the link. The first condition states that the shadow price for an arc equals the difference in nodal prices between its endpoints. The second and third set of conditions are called complementary slackness. They ensure that there are no profitable arbitrage trades in the radial network. Hence, nodes that are connected by uncongested arcs must have the same price. If two nodes are connected by a congested arc then the price at the importing end will be at least as large as the price in the exporting end. The fourth condition ensures that net-demand equals net-imports in every node. The fifth condition ensures that each producer is dispatched at a point where its marginal cost is equal to the local price.

Each arc is in one of three states depending on whether the flow is uncongested, at capacity in the positive direction, or at capacity in the reverse direction. Since our network has $K$ arcs there are $3^K$ different combinations of states for the arcs. We denote each of these combinations by an integer $\omega \in \Omega = \{1, 2, \ldots, 3^K\}$ called

a *congestion state.* For each congestion state $\omega$, and node $m$, we denote the set of nodes that are connected to $m$ by a chain of uncongested arcs by $\Xi_m(\omega)$. We say that nodes in this set are *completely integrated* with node $m$.

We note that it follows from the market clearing conditions that if the supply function $Q_{mn}(p_m)$ from producer $n$ in node $m$ results in the nodal price $p_m$, then any strictly increasing supply function $\hat{Q}_{mn}$ with $\hat{Q}_{mn}(p_m) = Q_{mn}(p_m)$ will also satisfy the market clearing conditions and result in the same dispatch and the same prices. This means that for given demand shock $\boldsymbol{\varepsilon}$ and supply functions of the competitors, any pair $\{p_m, Q_{mn}(p_m)\}$ with the dispatch of firm $n$ in node $m$ and the local price in node $m$ will be on the same curve, the residual demand curve. For a given realization of $\boldsymbol{\varepsilon}$ we introduce an *ex-post residual demand function* $D_{mn}^{\varepsilon}(p)$, which yields the dispatch for producer $n$ that would be observed in an optimal solution to DP if the producer is cleared at price $p$. Lemma 2 establishes properties of this function. But first we will prove that flows in the network increase away from node $m$ as the injected quantity in node $m$ increases. For this purpose, we define

$$
\begin{aligned}
\text{DP}(m,q): \quad \min \quad & \sum_{i \neq m} \int_0^{q_i} T_i(x)\, dx + \int_0^{q_{m,-n}} T_{m,-n}(x)\, dx \\
\text{s.t.} \quad & \mathbf{A t} = \boldsymbol{\varepsilon} - \mathbf{q}, \\
& -\bar{\mathbf{t}} \leq \mathbf{t} \leq \bar{\mathbf{t}}, \\
& q_m = q + q_{m,-n}.
\end{aligned}
$$

Note that the shadow price $p_m$ at node $m$ from the optimal solution to $\text{DP}(m,q)$ defines the inverse residual demand function faced by producer $n$. Our analysis of the residual demand curve has parallels to Downward et al (2010), where producers are restricted to make Cournot offers. In our case, we restrict the analysis to the case where $N_m \geq 2$, so that producer $n$ always faces some competition independent of the congestion state.where $T_{m,-n}(\cdot)$ is the inverse of $S_{m,-n}(\cdot)$.

**Lemma 1** *Suppose for node $m$ that $N_m \geq 2$. Then, in an optimal solution to DP(m,q), the flow away from node $m$ is nondecreasing with $q$ in every arc, and each congestion state is visited at most once.*

**Lemma 2** *Suppose for node $m$ that $N_m \geq 2$. Then for a fixed value of $\boldsymbol{\varepsilon}$ the ex-post residual demand curve of firm $n$ in node $m$ is a continuous, strictly decreasing function, $D_{mn}^{\varepsilon}(p_m)$, which is differentiable everywhere except possibly at a finite number of points where the congestion state changes. If the market is cleared away from such a point and the congestion state is $\omega$, then the residual demand function has derivative*

$$
D_{mn}'(p, \omega) := \frac{d}{dp} D_{mn}^{\varepsilon}(p) = -S_{m,-n}'(p) - \sum_{\ell \in \Xi_m(\omega) \setminus \{m\}} S_{\ell}'(p). \tag{4}
$$

Demand is inelastic, so for a given congestion state $\omega$, the slope of residual demand is given by the sum over competitors' supply function slopes in the integrated area $\Xi_m(\omega)$ consisting of nodes that are linked to node $m$ by uncongested lines in congestion state $\omega$.

We also introduce the function $\omega_{mn}(\boldsymbol{\varepsilon}, p)$, which gives the congestion state of the system for a given shock $\boldsymbol{\varepsilon}$ as producer $n$ in node $m$ moves along its residual

demand curve. At some prices $p$ the congestion state $\omega_{mn}(\boldsymbol{\varepsilon}, p)$ might switch as a line in the optimal dispatch becomes congested or uncongested. We call such a point a *kink* in $D^{\boldsymbol{\varepsilon}}_{mn}(p)$ at $p$. As shown in Lemma 2 above, $\frac{d}{dp} D^{\varepsilon}_{mn}(p)$ can change discontinuously at kinks.

## 2.3    Optimality conditions

In this subsection, we derive optimality conditions for the supply-function offer $Q_{mn}(\cdot)$ of firm $n$ in node $m$ of a radial network, given supply functions of its competitors in all nodes. The analysis is more complicated than in Klemperer and Meyer (1989), in the sense that a producer may not be able to predict the slope of its residual demand curve at a given price. Therefore a producer's output will often not be optimal ex-post, after the residual demand curve has been realized. Using Anderson and Philpott's (2002a) market-distribution-function approach, we will derive conditions that are optimal ex-ante, and that depend on the probability distribution of the shocks.

To derive the optimality conditions for a supply function $Q_{mn}(\cdot)$, we consider a candidate point $(p, q) \in (0, \bar{p}) \times (0, \bar{q}_{mn})$ with the nodal market price $p$ at node $m$ and the output $q$ of firm $n$, and investigate whether such a point could be part of an optimal supply function $Q_{mn}(\cdot)$. When testing for optimality at $(p, q)$, we restrict attention to the set of demand shocks $\boldsymbol{\varepsilon}$, for which the ex-post residual demand curve $D^{\boldsymbol{\varepsilon}}_{mn}(p_m)$ passes through or below the candidate point $(p, q)$. Thus we introduce the set

$$E_{mn}(p, q) = \{\boldsymbol{\varepsilon} \mid D^{\boldsymbol{\varepsilon}}_{mn}(p) \le q\}.$$

The probability for being in this set is equal to Anderson and Philpott's (2002a) market distribution function of the firm.

**Definition 1** *The* market distribution function *for firm $n$ at location $m$ is*

$$\psi_{mn}(p, q) = \mathbb{P}(D^{\boldsymbol{\varepsilon}}_{mn}(p) \le q) = \mathbb{P}(E_{mn}(p, q)).$$

$1 - \psi_{mn}(p, q)$ is equivalent to Wilson's "probability distribution of the sale price" in auctions of shares (Wilson, 1979).

In order to compute the market distribution function and its derivatives, we will partition $E_{mn}(p, q)$ into subsets that yield the same congestion states. To do this we need to establish some regularity properties of these subsets. These are a consequence of the continuity of solutions of the dispatch problem and the existence of a well-behaved density function for the demand shocks. We define the sets of demand shocks

$$\bar{E}_{mn}(p, q, \omega) = \{\boldsymbol{\varepsilon} \mid D^{\boldsymbol{\varepsilon}}_{mn}(p) \le q \text{ and } \omega_{mn}(\boldsymbol{\varepsilon}, p) = \omega\};$$

$$\partial E_{mn}(p, q, \omega) = \{\boldsymbol{\varepsilon} \mid D^{\boldsymbol{\varepsilon}}_{mn}(p) \le q, \omega_{mn}(\boldsymbol{\varepsilon}, p) = \omega \text{ and } (p, D^{\boldsymbol{\varepsilon}}_{mn}(p)) \text{ is a kink}\};$$

$$E_{mn}(p, q, \omega) = \bar{E}_{mn}(p, q, \omega) \setminus \partial E_{mn}(p, q, \omega).$$

As shown below, $\mathbb{P}(\partial E_{mn}(p, q, \omega)) = 0$, so when optimizing expected profit for player $n$, we can work with $E_{mn}(p, q, \omega)$ that disregards outcomes where $\omega$ is at

the boundary between two or more congestion states. This motivates the definition of a *state-dependent market distribution function*

$$\psi_{mn}(p, q, \omega) = \mathbb{P}(E_{mn}(p, q, \omega)).$$

**Lemma 3** $\mathbb{P}(\{\varepsilon \mid (p, D^{\varepsilon}_{mn}(p)) \text{ is a kink}\}) = \mathbb{P}(\partial E_{mn}(p, q, \omega)) = 0.$

Using Lemma 3 and the fact that the sets $E_{mn}(p, q, \omega)$ are disjoint we have

$$\psi_{mn}(p, q) = \sum_{\omega} \mathbb{P}(E_{mn}(p, q, \omega)) = \sum_{\omega} \psi_{mn}(p, q, \omega).$$

We define the conditional probability that the congestion state is $\hat{\omega}$ given that $D^{\varepsilon}_{mn}(p) = q$ by

$$\hat{P}_{mn}(\hat{\omega} \mid p, q) = \frac{\frac{\partial}{\partial q}\psi_{mn}(p, q, \hat{\omega})}{\sum_{\omega} \frac{\partial}{\partial q}\psi_{mn}(p, q, \omega)}. \tag{5}$$

To illustrate these constructions consider a simple network with two nodes connected by one arc from node 1 to node 2 with flow $t \in [-\bar{t}, \bar{t}]$ and demand shocks $\varepsilon_1$ and $\varepsilon_2$. There is a single producer $n$ in node 1 and producers with total supply function $S_2(p_2)$ in node 2. There are three congestion states: $\omega_1$ corresponds to $t \in (-\bar{t}, \bar{t})$, $\omega_2$ to $t = \bar{t}$, and $\omega_3$ to $t = -\bar{t}$. $E_{1n}(p, q, \omega)$ contains demand shocks for which the residual demand curve of firm $n$ is to the left of the point $(p, q)$ and $\omega_{1n}(\varepsilon, p) = \omega$. Figure 1 illustrates these sets for each congestion state. We see that $\psi_{1n}(p, q, \omega)$ is the measure of each shaded region $E_{1n}(p, q, \omega)$ to the left of the bold boundary, and $\frac{\partial}{\partial q}\psi_{mn}(p, q, \omega)$ gives the change in measure of each shaded region as $q$ changes. This can be interpreted as the line integral along the bold segment shown at the right-hand boundary of each shaded area.

The objective of a producer is to choose a supply function $Q_{mn}(\cdot)$ to maximize its expected profit. Given this function $Q_{mn}(\cdot)$, the fixed supply functions of other producers, and a demand shock $\varepsilon$, the dispatch problem DP yields a unique price $p_m(\varepsilon)$ at node $m$. The expected profit from $Q_{mn}(\cdot)$ is then

$$\int_{\mathcal{E}} (Q_{mn}(p_m(\varepsilon))p_m(\varepsilon) - C_{mn}(Q_{mn}(p_m(\varepsilon)))) f(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_M)d\varepsilon_1 d\varepsilon_2 \ldots d\varepsilon_M.$$

Observe that $p_m(\varepsilon)$ in this expression is endogenous; it depends on the choice of the supply function $Q_{mn}(\cdot)$. This makes maximizing expected profit appear to be very complicated since the expected profit has a different formula every time we choose a different $Q_{mn}(\cdot)$. As observed by Anderson and Philpott (2002a) the expected profit $\bar{\Pi}_{mn}$ for producer $n$ can be expressed in a way that does not require $p_m(\varepsilon)$ explicitly. Given a market distribution function $\psi_{mn}$ and an offer curve defined by $Q_{mn}(p)$, the expected profit of producer $m$ simplifies to

$$\bar{\Pi}_{mn} = \int (pQ_{mn}(p) - C_{mn}(Q_{mn}(p))) \left(\frac{\partial \psi_{mn}}{\partial p} + Q'_{mn}(p)\frac{\partial \psi_{mn}}{\partial q}\right)dp,$$
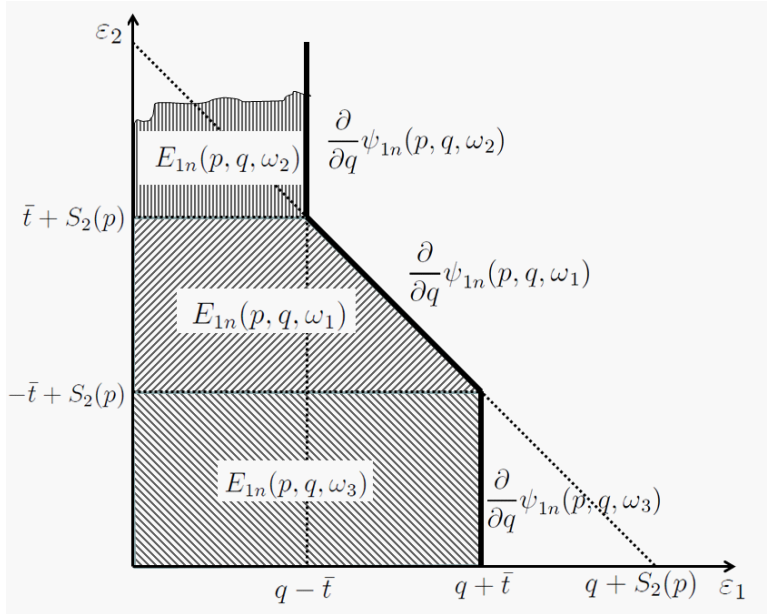
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Figure 1: Computation of $\frac{\partial}{\partial q}\psi_{1n}(p,q,\omega)$ for a two-node network with $S_{1,-n}(p) = 0$. The different shadings show the regions $E_{1n}(p,q,\omega)$. The values of $\frac{\partial}{\partial q}\psi_{1n}(p,q,\omega)$ are integrals along the right-hand boundary of each of these regions as shown.

which is a one-dimensional integral.

We are now ready to introduce the optimality condition. We will express the optimality condition in terms of a function $Z$, where $Z = 0$ corresponds to the Euler equation. Our definition of $Z$ is analogous to that introduced by Anderson and Philpott (2002a), but in our case it is specific for radial networks, and should be multiplied by $\partial\psi_{mn}(\hat{p},\hat{q})/\partial q$ to make it consistent with the $Z$ function that was originally introduced by Anderson and Philpott (2002a). This rescaling of the $Z$-function is somewhat easier to work with in our context, but, unlike the $Z$ function in Anderson and Philpott (2002a), our version cannot be applied to circumstances where supply functions have vertical or horizontal segments due to binding monotonicity constraints.

**Definition 2**

$$Z_{mn}(p,q) = (p - C'_{mn}(q)) \sum_{\omega} -D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \mid p,q) - q. \tag{6}$$

The sum $\sum_{\omega} -D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \mid p,q)$ in (6) is the expected slope in the residual demand that firm $n$ is facing at a point $(p,q)$, in other words

$$\mathbb{E}_{\omega}\left[-D'_{mn}(p,\omega) \mid D^{\varepsilon}_{mn}(p) = q\right] = \sum_{\omega} -D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \mid p,q). \tag{7}$$

This can be interpreted as a quantity effect, i.e. how many units are lost in expectation when producer $n$ increases the local price in node $m$ by one unit, conditional on the residual demand passing through the point $(p,q)$. Multiplying

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the lost quantity by the mark-up $(p - C'_{mn}(q))$ gives the lost value due to the quantity effect. The second term, $q$, in (6) is the price effect. This is what producer $n$ would gain in expectation from increasing the price in node $m$ by one unit if its output is fixed at $q$. There is an extremum when the price effect equals the lost value due to the quantity effect, so that $Z_{mn}(p, q) = 0$. The extremum is a profit maximum if the quantity effect dominates the price effect at prices above the extremum price and if the price effect dominates at prices below the extremum price. For a monotonic increasing supply function $Q_{mn}(p)$ this is equivalent to the following statement, which is formally proven in Appendix A by means of results in Anderson and Philpott (2002a).

**Proposition 1** *In radial networks, a monotonic increasing supply function $Q_{mn}(p)$ is globally optimal if it satisfies*

$$\begin{cases} Z_{mn}(p, q) \geq 0, & \text{if } q < Q_{mn}(p) \\ Z_{mn}(p, q) = 0, & \text{if } q = Q_{mn}(p) \\ Z_{mn}(p, q) \leq 0, & \text{if } q > Q_{mn}(p). \end{cases} \tag{8}$$

The following necessary first-order condition follows from Definition 2 and $Z_{mn}(p, q) = 0$.

**Corollary 1** *For radial networks, a monotonic increasing optimal supply function $Q_{mn}(p_m)$ satisfies*

$$Q_{mn}(p_m) = (p_m - C'_{mn}(Q_{mn}(p_m)))\, \mathbb{E}_\omega \left[ -D'_{mn}(p_m, \omega) \mid D^\varepsilon_{mn}(p_m) = Q_{mn}(p_m) \right]. \tag{9}$$

This generalizes the first-order condition of Klemperer and Meyer (1989) to multi-dimensional shocks, by saying that the optimal output of a producer at its local price $p_m$ is proportional to its mark-up and the expected slope of the residual demand that it is facing at $p_m$. Wilson (2008) derives a similar first-order condition. The necessary first-order condition could be used to empirically test the optimal bidding behaviour of producers in the presence of transmission congestion. $\mathbb{E}_\omega \left[ -D'_{mn}(p_m, \omega) \mid D^\varepsilon_{mn}(p_m) = Q_{mn}(p_m) \right]$ would then be estimated from the historical average slope of residual demand at price $p_m$ when the output of firm $n$ in node $m$ is $Q_{mn}(p_m)$. Previous empirical studies of bidding behaviour in wholesale electricity markets (e.g. Sioshansi and Oren, 2007; Hortaçsu and Puller, 2008; Wolak, 2007) have neglected transmission constraints.

With minor edits in the proofs, it can be shown that the necessary and sufficient optimality conditions would hold also for strategic producers competing with supply functions in networks with shocks in the transmission capacities, a case that is considered by Wilson (2008). In the analysis of capacity constrained networks, the state normally depends on the price. A simpler structure of the stochastic residual demand is when the state of the ex-post residual demand curve would not depend on the price. For example, our necessary condition in (9) would also hold when there is a finite number of continuous and decreasing residual-demand-function types (states) that can occur and where each such function type is shifted by an additive shock.

## 2.4 Computing $\psi_{mn}(p, q, \omega)$

In order to compute SFE in transmission networks, we need to compute $\frac{\partial}{\partial q}\psi_{mn}(p, q, \omega)$ for each congestion state $\omega$. This involves determining a market outcome for every realization of the vector $\boldsymbol{\varepsilon}$, and then integrating the multivariate density function $f$ over the volume in $\boldsymbol{\varepsilon}$-space that corresponds to events where firm $n$ in node $m$ sells $q$ units at price $p$ and that the system is in congestion state $\omega$. In the general case this volume is complicated. Similar to Wilson (2008), we avoid this by transforming the problem into one where we instead integrate over the flows and shadow prices that arise in each congestion state. In these calculations, we find it useful to denote by $L(\omega)$, $B(\omega)$, and $U(\omega)$ the sets of arcs where flows are at their lower bound (i.e. congested in the negative direction), between their bounds or at their upper bound, respectively. Hence, the complementary slackness conditions, i.e. the second and third set of conditions in (3), can be equivalently written as follows:

$$
\begin{array}{llll}
t_k = \bar{t}_k, & \sigma_k = 0, & \rho_k > 0, & k \in U(\omega), \\
t_k \in (-\bar{t}_k, \bar{t}_k) & \sigma_k = 0, & \rho_k = 0, & k \in B(\omega), \\
t_k = -\bar{t}_k, & \sigma_k > 0, & \rho_k = 0, & k \in L(\omega).
\end{array}
$$

Observe that given a congestion state $\omega$ and arc $k$, there is exactly one variable $t_k$, $\rho_k$ or $\sigma_k$ that is not at a bound.

We partition $\mathbf{t}$ and the shadow prices $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ into $(\mathbf{t}_L, \mathbf{t}_B, \mathbf{t}_U)$, $(\boldsymbol{\sigma}_L, \mathbf{0}_B, \mathbf{0}_U)$ and $(\mathbf{0}_L, \mathbf{0}_B, \boldsymbol{\rho}_U)$ corresponding to flows at their lower bounds, strictly between their bounds, and at their upper bounds. Below we define the volume $\mathcal{T}(B(\omega))$ in $\mathbf{t}$ space that the flows in the set of uncongested arcs $B(\omega)$ can span. $\mathcal{U}(U(\omega))$ and $\mathcal{L}(L(\omega))$ are the volumes in $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space spanned by the shadow prices of congested arcs in the sets $U(\omega)$ and $L(\omega)$, respectively. Recall these volumes are all open sets, as we neglect outcomes at the boundary between two congestion states.

$$
\begin{array}{l}
\mathcal{T}(B(\omega)) = \{\mathbf{t}_{B(\omega)} : -\bar{\mathbf{t}}_{B(\omega)} < \mathbf{t}_{B(\omega)} < \bar{\mathbf{t}}_{B(\omega)}\}, \\
\mathcal{U}(U(\omega)) = \{\boldsymbol{\rho}_{U(\omega)} : \mathbf{0} < \boldsymbol{\rho}_{U(\omega)}\}, \\
\mathcal{L}(L(\omega)) = \{\boldsymbol{\sigma}_{L(\omega)} : \mathbf{0} < \boldsymbol{\sigma}_{L(\omega)}\}.
\end{array}
\tag{10}
$$

In particular we are interested in $\mathcal{S}(\omega) \subseteq \mathbb{R}^K$, which we define by

$$
\mathcal{S}(\omega) = \mathcal{L}(L(\omega)) \times \mathcal{U}(U(\omega)) \times \mathcal{T}(B(\omega)).
\tag{11}
$$

Hence, $\mathcal{S}(\omega)$ is the total volume in $\mathbf{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space that is spanned for a congestion state $\omega$.

We introduce additional notation in order to analyse the radial network in detail. For each state $\omega$, we partition the nodes into the sets $\Xi_m(\omega)$ and $F(\omega)$, where as before $\Xi_m(\omega)$ includes all nodes that are connected to node $m$ through some uncongested chain of arcs. The set $F(\omega)$ contains all other nodes in the network. Similarly we partition the shock vector into $\boldsymbol{\varepsilon}_{\Xi_m(\omega)}$ and $\boldsymbol{\varepsilon}_{F(\omega)}$. Let $\kappa(\omega)$ be the set of uncongested arcs that connect nodes in $\Xi_m(\omega)$. Other arcs are in the set $\vartheta(\omega)$. We let $\mathbf{t}_{\kappa(\omega)}$ be the flows in the uncongested arcs between nodes in the set $\Xi_m(\omega)$ and we let $\mathbf{t}_{\vartheta(\omega)}$ be the vector of flows in the other arcs.

**Lemma 4**

$$\frac{\partial \psi_{mn}(p, q, \omega)}{\partial q} = \int\limits_{\mathcal{S}(\omega)} f\left(\mathbf{At} + \mathbf{s}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q\right)\right) J(\omega) \mathbf{dt}_{B(\omega)} \mathbf{d}\boldsymbol{\rho}_{U(\omega)} \mathbf{d}\boldsymbol{\sigma}_{L(\omega)}, \quad (12)$$

*where*

$$\begin{aligned}
\mathbf{s}_{-m}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q\right) &= \mathbf{s}_{-m}\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma})\right) \\
s_m\left(\mathbf{p}(p, \boldsymbol{\rho}, \boldsymbol{\sigma}), q\right) &= q + S_{m,-n}(p)
\end{aligned}$$

*and*

$$J(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial\left(\left(\mathbf{t}_{\vartheta(\omega)}\right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}\right)} \right|. \quad (13)$$

Lemma 9 in the Appendix shows how $J(\omega)$ can be computed.

## 2.5 Conditions for symmetric equilibria

Our necessary and sufficient optimality conditions can in principle be used to solve for asymmetric SFE. However, we will focus on solving for symmetric equilibria in this paper. In this case, we find it useful to introduce a market integration function as below.

**Definition 3** *For firm $n$ in node $m$ we define the* market integration function *by*

$$\mu_{mn}(p, q) = \sum_\omega M_{\Xi_m(\omega)} \hat{P}_{mn}(\omega | p, q),$$

*where $M_{\Xi_m(\omega)}$ is the number of nodes in the set $\Xi_m(\omega)$.*

Thus, the market integration function is equal to the expected number of nodes (including node $m$ itself) that are completely integrated with node $m$ given that firm $n$ has output $q$ and node $m$ has the market price $p$. In a network that is symmetric with respect to producers and where producers submit identical supply functions, every producer would have the same market integration function $\mu(p, q)$. For a given inverse nodal supply function $T_m(q)$, we can also define a market integration function with respect to the nodal output $q$

$$\widetilde{\mu}(q) = \mu\left(T_m(q), q/N\right).$$

Below we show that $\widetilde{\mu}(q)$ does not depend on the function $T_m(q)$. The reason is that demand is inelastic in our model, so the number of production units that are needed to meet a given demand shock outcome does not depend on market competition. Moreover, the order in which production units of symmetric firms are accepted is the same irrespective of their symmetric mark-ups. Thus even if producers have market power and use it, market integration can be determined from a simple model with price-taking producers, where $T_m(q) = C'(q/N)$.

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**Lemma 5** *Consider a network that is symmetric with respect to producers. Each strategic node has $N$ identical producers with production costs $C(Q)$, and each producer chooses a supply function $Q(p)$ in order to maximize its expected profit. A necessary condition for $Q(p)$ to be a SFE is*

$$Q = (p - C'(Q))\left(\widetilde{\mu}(NQ(p))N - 1\right)Q', \tag{14}$$

*where $\widetilde{\mu}(q)$ is independent of $T_m(q)$.*

It follows from (14) that symmetric oligopoly producers will increase their mark-ups at output levels where the (exogenous) market integration function $\widetilde{\mu}(NQ)$ is small, i.e. when arcs to node $m$ are congested with a high conditional probability. Similarly, oligopoly producers will decrease their mark-ups at output levels where the market integration function $\widetilde{\mu}(NQ)$ is large.

# 3 Examples

We can use the first-order condition $Z_{mn}(p, q) = 0$ to construct a first-order condition for each firm in a radial network. The SFE can be solved from a system of such first-order conditions for general radial networks. The global second-order condition of an available first-order solution can be verified by (8). In this section we use these optimality conditions to derive SFE for well-behaved two-node and star networks with symmetric firms.

To ensure that there is a unique Nash equilibrium in the examples, we assume that producer $n$ in node $m$ has a capacity constraint $\bar{q}_{mn}$. Moreover, we introduce a reservation price $\bar{p}$ (price cap) such that $\bar{p} > C'_{mn}(\bar{q}_{mn})$ for all $m \in \{1, \ldots, M\}$ and all $n \in \{1, \ldots, N_m\}$. We also impose example-specific restrictions on the support of the demand shocks to make sure that there is always a feasible dispatch. The support is as wide as possible to avoid ambiguities. If it would be the case that a producer would never be cleared along a segment of its supply function, then the shape of this segment would not be determined by profit maximizing behaviour, and this would normally lead to multiplicity of equilibria.

## 3.1 Well-behaved two-node network

Recall the simple network with two nodes connected by one arc from node 1 to node 2 with flow $t \in [-\bar{t}, \bar{t}]$. There are three congestion states: $\omega_1$ corresponds to $t \in (-\bar{t}, \bar{t})$, $\omega_2$ to $t = \bar{t}$, and $\omega_3$ to $t = -\bar{t}$. We derive the optimality condition for a firm in node 1 with price $p = p_1$. It can be shown that:

**Lemma 6** *In a two-node network, the optimal supply function of firm $n$ in node 1 can be determined from*

$$\begin{aligned}
Z_{1n}(p, q) = {} & (p - C'_{1n}(q))(S'_{1,-n}(p) + S'_2(p))\hat{P}(\omega_1 \mid p, q) \\
& + (p - C'_{1n}(q))S'_{1,-n}(p)\left(\hat{P}(\omega_2 \mid p, q) + \hat{P}(\omega_3 \mid p, q)\right) - q = 0,
\end{aligned} \tag{15}$$

where $\hat{P}(\omega \mid p, q)$ is defined by (5) from

$$
\begin{aligned}
\frac{\partial \psi_{1n}(p,q,\omega_1)}{\partial q} &= \int_{-\bar{t}}^{\bar{t}} f\left(q + S_{1,-n}(p) - t, S_2(p) + t\right) dt \\
\frac{\partial \psi_{1n}(p,q,\omega_2)}{\partial q} &= \int_{S_2(p)+\bar{t}}^{\infty} f\left(q + S_{1,-n}(p) - \bar{t}, \varepsilon_2\right) d\varepsilon_2 \\
\frac{\partial \psi_{1n}(p,q,\omega_3)}{\partial q} &= \int_{-\infty}^{S_2(p)-\bar{t}} f\left(q + S_{1,-n}(p) + \bar{t}, \varepsilon_2\right) d\varepsilon_2.
\end{aligned}
\tag{16}
$$

Below we consider symmetric Nash equilibrium for a two-node network that is symmetric with respect to producers. The existence of an equilibrium depends on the partial derivatives $f_m(\varepsilon_1, \varepsilon_2) = \frac{\partial f(\varepsilon_1, \varepsilon_2)}{\partial \varepsilon_m}$, $m = 1, 2$, of the shock density which must be sufficiently small. In our analysis of the two-node example, we require $f$ to have support defined by the convex region

$$
\mathcal{E}_2 = \left\{ (\varepsilon_1, \varepsilon_2) : \begin{array}{l} 0 \leq \varepsilon_1 + \varepsilon_2 \leq 2N\bar{q}, \\ -\bar{t} \leq \varepsilon_1 \leq N\bar{q} + \bar{t}, \\ -\bar{t} \leq \varepsilon_2 \leq N\bar{q} + \bar{t}, \end{array} \right\}
$$

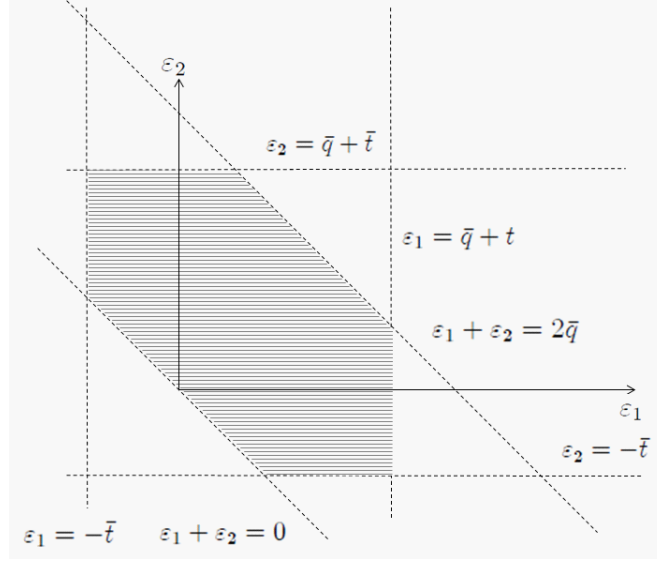as shown in Figure 2. It can be shown that symmetric solutions to (14) are



Figure 2: Well behaved two-node example. Shaded region depicts $\mathcal{E}_2$ the support of $f(\varepsilon_1, \varepsilon_2)$.

equilibria under the following circumstances.

**Proposition 2** *Consider a two-node network with $N$ symmetric firms in each node, each firm having identical production capacities $\bar{q}$ and identical marginal costs that are either constant or strictly increasing. If demand has a bounded shock density that satisfies $f(\varepsilon_1, \varepsilon_2) = f(\varepsilon_2, \varepsilon_1) > 0$ and $2N\bar{q}|f_m(\varepsilon_1, \varepsilon_2)| \leq (3N - 2) f(\varepsilon_1, \varepsilon_2)$ when $(\varepsilon_1, \varepsilon_2) \in \mathcal{E}_2$, then there exists a unique symmetric SFE in the network. Each firm's monotonic equilibrium offer, $Q(p)$, can be calculated*

*for $p \in (C'(0), \overline{p}]$ from the initial condition $Q(\overline{p}) = \overline{q}$ and*

$$Q'(p) = \frac{Q(p)}{(p - C'(Q(p)))(N\widetilde{\mu}(NQ(p)) - 1)}, \tag{17}$$

$$\widetilde{\mu}(NQ) = 1 + \frac{h(NQ, \omega_1)}{\sum_\omega h(NQ, \omega)}. \tag{18}$$

*The functions $h(NQ, \omega)$ are given by:*

$$h(NQ, \omega_1) = \int_{-\overline{t}}^{\overline{t}} f(NQ - t, NQ + t)\, dt,$$

$$h(NQ, \omega_2) = \int_{NQ+\overline{t}}^{N\overline{q}+\overline{t}} f(NQ - \overline{t}, \varepsilon_2)\, d\varepsilon_2, \tag{19}$$

$$h(NQ, \omega_3) = \int_{-\overline{t}}^{NQ-\overline{t}} f(NQ + \overline{t}, \varepsilon_2)\, d\varepsilon_2.$$

Proposition 2 ensures existence of equilibria when slopes in the probability density of the demand shocks are sufficiently small, which is a new contribution. In previous studies, which assume certain demand, existence of pure-strategy Nash equilibria in transport-constrained networks has only been established for cases where transport constraints are either far from binding or firmly binding (Borenstein et al., 2000). With certain demand, producers would typically have profitable deviations from first-order solutions when transport constraints are close to binding. In our case, where demand is uncertain we find a pure-strategy NE although the transmission-line can be binding, non-binding or close to binding with positive probabilities.

In the next step we will explicitly solve for the unique symmetric SFE in the two-node network. To simplify the optimality conditions we consider the case where demand shocks follow a bivariate uniform distribution.

**Assumption 1:** *Consider a network with two nodes connected by an arc with capacity $\overline{t}$ and with $N$ symmetric firms in each node. Inelastic demand in node $m \in \{1, 2\}$ is given by the shock $\varepsilon_m$. We assume that shocks are uniformly distributed with a constant density over the convex region $\mathcal{E}_2$ and zero elsewhere.*

**Proposition 3** *Under Assumption 1, the symmetric market integration function for the two-node network is given by*

$$\mu = \frac{4\overline{t} + N\overline{q}}{2\overline{t} + N\overline{q}}. \tag{20}$$

*There is a unique symmetric SFE with inverse symmetric supply functions that can be calculated from*

$$p(Q) = Q^{-1}(Q) = \frac{\overline{p}Q^{\mu N - 1}}{\overline{q}^{\mu N - 1}} + (\mu N - 1)Q^{\mu N - 1}\int_Q^{\overline{q}} \frac{C'(u)\, du}{u^{\mu N}}. \tag{21}$$

It follows from Proposition 3 that the market integration function $\mu$ simplifies to a constant for uniformly distributed demand shocks. In this case, the equilibrium offer of a firm in the two-node network with $N$ symmetric firms per node is identical to the equilibrium offer of a firm in an isolated node with $\mu N$ symmetric firms. We note that the market integration function $\mu$ is close to 2 when the transmission capacity $\bar{t}$ is significantly larger than the nodal production capacity $N\bar{q}$, so that the two nodes are almost completely integrated. In the other extreme when the transmission capacity is much smaller than the nodal production capacity, then the market integration factor is close to 1, i.e. the two markets are almost isolated from each other, so that a node is approximately only integrated with itself. As the equations are identical to the single node case in Holmberg (2008), solutions to (21) have the following properties:

1. Mark-ups are positive for a positive output.

2. For a given nodal production cost function, mark-ups decrease at every positive nodal output level with more symmetric firms in the market.

## 3.2   Badly-behaved two-node networks

Existence of NE is problematic for steep slopes in the shock density and especially so when it has discontinuities. This is illustrated by the non-existence example below.

**Example 1** *Shock densities with discontinuities: Assume that the probability density $f(\varepsilon_1, \varepsilon_2)$ is differentiable inside the support $[0, \bar{\varepsilon}] \times [0, \bar{\varepsilon}]$, but decreases discontinuously to zero when $\varepsilon_1 = \bar{\varepsilon}$ and $\varepsilon_2 \in [0, \bar{\varepsilon}]$, where*

$$\bar{t} < \bar{\varepsilon} < \bar{q} + \bar{t}. \tag{22}$$

*Thus the maximum demand shock is sufficiently large to congest the line, provided the output at the importing node is sufficiently small. However, the maximum demand shock is not large enough to exhaust both the import capacity and local production capacity. Thus assumptions in Proposition 2 are violated. Consider a potential symmetric NE of a duopoly market with one firm in each node with identical costs $C(q)$ and identical supply functions $Q(p)$. Assume that the symmetric supply functions $Q(p)$ are monotonic. In the following we will show that the producer in node 1 will have a profitable deviation from the potential symmetric pure-strategy NE. In particular we will consider the point $(q_0, p_0)$, where*

$$q_0 = Q(p_0) = \bar{\varepsilon} - \bar{t} \in (0, \bar{q}), \tag{23}$$

*because of the inequality in (22). There is only one firm per node in our example, so $S_2(p) = Q(p)$ and $S_{1,-n}(p) = 0$. It now follows from (16) that*

$$\frac{\partial}{\partial q}\psi_{1n}(p, q, \omega_3) = \int_{-\infty}^{Q(p)-\bar{t}} f(q + \bar{t}, \varepsilon)d\varepsilon$$

where $\omega_3$ is the congestion state where imports to node 1 are congested. Since $f(q + \bar{t}, \varepsilon)$ is positive when $q + \bar{t} < \bar{\varepsilon}$ and zero when $q + \bar{t} > \bar{\varepsilon}$ we have

$$\lim_{q\uparrow\bar{\varepsilon}-\bar{t}}\frac{\partial}{\partial q}\psi_{1n}(p,q,\omega_3) > \lim_{q\downarrow\bar{\varepsilon}-\bar{t}}\frac{\partial}{\partial q}\psi_{1n}(p,q,\omega_3) = \lim_{q\downarrow\bar{\varepsilon}-\bar{t}}\int_{-\infty}^{Q(p)-\bar{t}}f(q+\bar{t},\varepsilon)d\varepsilon = 0, \quad (24)$$

as illustrated in Figure 3. Thus at the point $(p_0, q_0)$ of its supply function $Q(p)$,
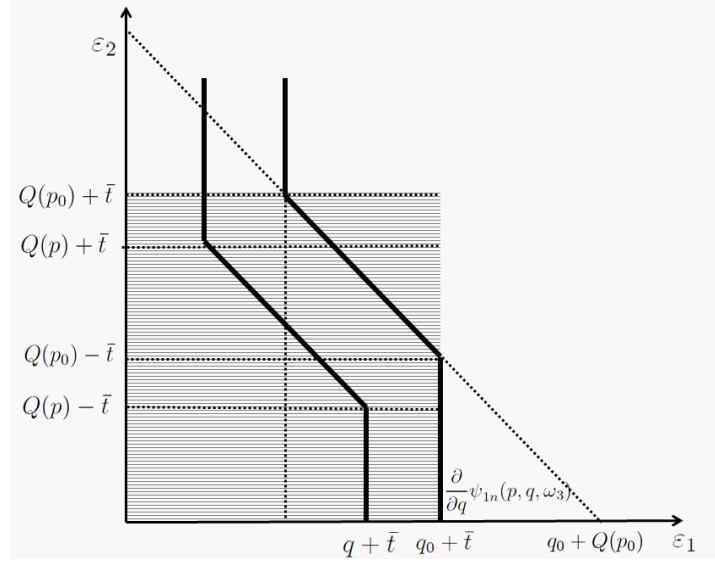


Figure 3: Demand shock density in Example 1 has support shown by shaded region. It is discontinuous when $\varepsilon_1 = \bar{\varepsilon} = q_0 + \bar{t}$. This yields a profitable deviation for producer $n$ to withhold just below $q_0$ and congest the line (state $\omega_3$) with non-neglible probability.

the producer in node 1 can discontinuously increase the probability that imports are congested by slightly withholding output. However, $\frac{\partial}{\partial q}\psi_{1n}(p,q,\omega_1)$ (the line is uncongested) and $\frac{\partial}{\partial q}\psi_{1n}(p,q,\omega_2)$ (exports are congested) are still continuous at the point $(q_0, p_0)$. From (16) we have:

$$\frac{\partial}{\partial q}\psi_{1n}(p_0,q_0,\omega_1) = \int_{-\bar{t}}^{\bar{t}}f(q_0-t,q_0+t)dt = \int_{-\bar{t}}^{\bar{t}}f(\bar{\varepsilon}-\bar{t}-t,\bar{\varepsilon}-\bar{t}+t)dt > 0$$

$$\frac{\partial}{\partial q}\psi_{1n}(p_0,q_0,\omega_2) = \int_{\bar{t}+q_0}^{\infty}f(q_0-\bar{t},\varepsilon)d\varepsilon = \int_{\bar{\varepsilon}}^{\infty}f(\bar{\varepsilon}-2\bar{t},\varepsilon)d\varepsilon = 0,$$

so (24) implies that

$$\lim_{q\downarrow\bar{\varepsilon}-\bar{t}}\sum_{\omega}\frac{\partial}{\partial q}\psi_{1n}(p_0,q,\omega) < \lim_{q\uparrow\bar{\varepsilon}-\bar{t}}\sum_{\omega}\frac{\partial}{\partial q}\psi_{1n}(p_0,q,\omega)$$

and accordingly

$$\lim_{q\downarrow\bar{\varepsilon}-\bar{t}}\hat{P}(\omega_1 \mid p_0,q) = \lim_{q\downarrow\bar{\varepsilon}-\bar{t}}\frac{\frac{\partial}{\partial q}\psi_{1n}(p_0,q,\omega_1)}{\sum_{\omega}\frac{\partial}{\partial q}\psi_{1n}(p_0,q,\omega)}$$

$$> \lim_{q\uparrow\bar{\varepsilon}-\bar{t}}\frac{\frac{\partial}{\partial q}\psi_{1n}(p_0,q,\omega_1)}{\sum_{\omega}\frac{\partial}{\partial q}\psi_{1n}(p_0,q,\omega)} = \lim_{q\uparrow\bar{\varepsilon}-\bar{t}}\hat{P}(\omega_1 \mid p_0,q). \quad (25)$$

As shown by Anderson and Philpott (2002a) a necessary condition for $(p_0, q_0)$ to be on an optimal response curve for producer $n$ is $\lim_{q \uparrow \bar{\varepsilon} - \bar{t}} Z_{1n}(p_0, q) \geq 0$ and $\lim_{q \downarrow \bar{\varepsilon} - \bar{t}} Z_{1n}(p_0, q) \leq 0$. However (15) and (25) implies that

$$
\begin{aligned}
\lim_{q \downarrow \bar{\varepsilon} - \bar{t}} Z_{1n}(p_0, q) &= (p - C'(q_0))Q'(p_0) \lim_{q \downarrow \bar{\varepsilon} - \bar{t}} \hat{P}(\omega_1 \mid p_0, q) - q_0 \\
&> (p - C'(q_0))Q'(p_0) \lim_{q \uparrow \bar{\varepsilon} - \bar{t}} \hat{P}(\omega_1 \mid p_0, q) - q_0 \\
&= \lim_{q \uparrow \bar{\varepsilon} - \bar{t}} Z_{1n}(p_0, q),
\end{aligned}
$$

which is a contradiction. Thus there is a profitable deviation from $Q(p)$.

Assumptions in Proposition 2 are also violated if demand shocks in the two nodes are sufficiently correlated. In particular, existence of symmetric pure-strategy NE is ruled out in an example by Holmberg and Philpott (2015), the working paper version of this article, when demand shocks are perfectly correlated. In such an extreme case, a producer would be able to infer the slope of its residual demand curve from the market price and thereby locate at what price convex kinks would occur. Similar to the incentives to congest for certain demand analysed by Borenstein et al. (2000), perfectly correlated shocks give a producer in a node where imports are nearly congested the incentive to unilaterally deviate from the first-order solution by withholding power in order to congest imports so as to increase the price of the importing node.

## 3.3 Well-behaved star network

Next, we consider a star network with four nodes and three radial lines with capacity $\bar{t}$ as shown in Figure 4. Firms are located in nodes $1 - 3$ and each arc has the same number as the starting node, i.e. $1, 2$ or $3$.
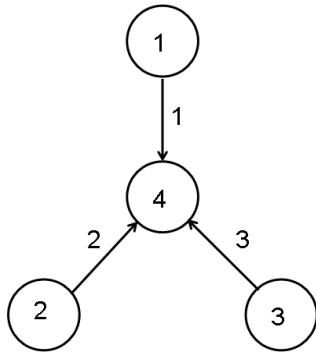


Figure 4: Star network example.

Demand shocks with density $\frac{1}{V}$ are defined on the region

$$
\mathcal{E}_4 = \left\{
\begin{array}{c}
(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4) \in \mathbb{R}^4 \mid -\bar{t} \leq \varepsilon_j \leq N\bar{q} + \bar{t}, \ -3\bar{t} \leq \varepsilon_4 \leq 3\bar{t}, \\
-2\bar{t} \leq \varepsilon_j + \varepsilon_4 \leq N\bar{q} + 2\bar{t}, \ -\bar{t} \leq \varepsilon_j + \varepsilon_\ell + \varepsilon_4 \leq 2N\bar{q} + \bar{t}, \\
0 \leq \varepsilon_1 + \varepsilon_2 + \varepsilon_3 + \varepsilon_4 \leq 3N\bar{q}, \\
\forall j \in \{1, 2, 3\} \ \text{and} \ \forall \ell \in \{1, 2, 3\}, \ \text{where} \ \ell \neq j
\end{array}
\right\}
$$

where we let $V$ be the volume of $\mathcal{E}_4$.

**Assumption 2.** *There are $N$ firms with identical costs $C(q)$ and identical production capacities $\overline{q}$ in each strategic node $1-3$. There are no producers in node 4 (the center node). Inelastic demand in node $j \in \{1, 2, 3, 4\}$ is given by $\varepsilon_j$. Demand shocks are uniformly distributed such that:*

$$f(\boldsymbol{\varepsilon}) = \begin{cases} \frac{1}{V} & \text{if } \boldsymbol{\varepsilon} \in \mathcal{E}_4 \\ 0 & \text{otherwise.} \end{cases}$$

Thus the shock density and network are symmetric with respect to the strategic nodes $1, 2, 3$. We can show the following under these circumstances:

**Proposition 4** *Under Assumption 2, the symmetric market-integration function is a constant given by:*

$$\mu = \frac{3(N\overline{q})^2 + 12\overline{t}N\overline{q} + 12\overline{t}^2}{3(N\overline{q})^2 + 8\overline{t}N\overline{q} + 4\overline{t}^2}. \tag{26}$$

*There is a unique symmetric SFE with inverse symmetric supply functions that can be calculated from (21).*

# 4 Conclusions

We derive optimality conditions for supply functions of producers competing in a network with transport constraints and local demand shocks. We show that the optimal output of a producer is proportional to its mark-up and the expected slope of its residual demand curve at every local price of the producer. In principle, a system of such optimality conditions can be used to numerically calculate asymmetric supply-function equilibria (SFE) in a radial networks. A Working Paper version of this paper, Holmberg and Philpott (2015), present optimality conditions for meshed networks.

In this paper, we characterize symmetric SFE in radial networks with inelastic demand. We verify that there is a unique symmetric monotonic solution to the first-order condition and that this solution is an SFE in a two-node network when the joint probability density of the local demand shocks is sufficiently evenly distributed, i.e. sufficiently close to a uniform multi-dimensional distribution. But existence of SFE cannot be taken for granted. Profitable deviations from the first-order solution will for example exist for perfectly correlated demand shocks or steep slopes and discontinuities in the demand shock density.

For symmetric equilibria in radial networks with inelastic demand, it is useful to define a market integration function, which equals the expected number of nodes that are completely integrated with the node of the producer under study. Firms' mark-ups depend on the number of firms in the market. Still it can be shown that in a symmetric equilibrium, market integration is a function of the total production in a node. This function can be determined from exogenous parameters for price-taking producers. The implication is that oligopoly producers will have high mark-ups at output levels for which the (exogenous) market integration function returns

small values, and lower mark-ups at output levels where market integration is expected to be high.

The market integration function simplifies to a constant for symmetric equilibria in radial networks with multi-dimensional uniformly distributed shocks. In this case, we use our optimality conditions to explicitly solve for symmetric equilibria in two-node and star networks. We also show that these symmetric equilibria are well-behaved: (i) mark-ups are positive for a positive output, and (ii) for a given total production cost, mark-ups decrease with more firms in the market.

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# Appendix A: Proofs for Section 2

## Market clearing conditions

**Proof. (Lemma 1)** Assume for the purposes of this proof that all arcs are directed away from node $m$. By Berge's theorem of the maximum and strict concavity of the objective function, the vector $\mathbf{t}$ of arc flows in an optimal solution of $\mathrm{DP}(m, q)$ is a continuous function of $q$, and yields a unique congestion state $\omega(q)$. Given an optimal dispatch and prices $\mathbf{p}$, the flow balance condition at node $m$ for this optimal solution is

$$q + S_{m,-n}(p_m) + \sum_{j \in \Xi_m(\omega(q)) \setminus \{m\}} S_j(p_m) = \sum_{j \in \Xi_m(\omega(q))} \varepsilon_j + \sum_{k \in \delta^+(\omega(q),m)} \bar{t}_k - \sum_{k \in \delta^-(\omega(q),m)} \bar{t}_k \tag{27}$$

where $\sum_{k \in \delta^+(\omega(q),m)} \bar{t}_k$ adds up the congested exports to the network from the integrated area $\Xi_m(\omega(q))$, and $\sum_{k \in \delta^-(\omega(q),m)} \bar{t}_k$ adds up the congested imports from the network into $\Xi_m(\omega(q))$. In congestion state $\omega(q)$, the supply $S_{m,-n}(p_m) + \sum_{j \in \Xi_m(\omega(q)) \setminus \{m\}} S_j(p_m)$ must decrease as $q$ increases, so $p_m$ decreases as well.

In congestion state $\omega(q)$ the flows in all arcs in $\delta^-(\omega(q), m) \cup \delta^+(\omega(q), m)$ remain the same, as do the flows in arcs that join nodes outside $\Xi_m(\omega(q))$. So within congestion state $\omega(q)$ consider an uncongested arc $k$ directed from node $r$ to node $\ell$ in $\Xi_m(\omega(q))$. We suppose that $t_k$ decreases with increasing $q$ and derive a contradiction. Let $\Gamma_\ell(\omega(q))$ denote the nodes in the subtree rooted at node $\ell$ obtained by removing arc $k$ from the tree spanning $\Xi_m(\omega(q))$. If the congestion state remains the same and $t_k$ decreases then (since $p_m$ decreases as well) the total supply into nodes in $\Gamma_\ell(\omega(q))$ is strictly less than $\sum_{j \in \Gamma_\ell(\omega(q))} \varepsilon_j$ and so $\mathrm{DP}(m, q)$ is infeasible, which is a contradiction. Thus all flows are nondecreasing with $q$ while congestion state $\omega(q)$ remains constant. Recall that arc flows vary continuously with $q$, even when the congestion state is switching. It follows that all arc flows are nondecreasing with $q$.

Finally, since changing from one congestion state to another must strictly increase the flow in at least one arc, each congestion state can be visited at most once as $q$ increases. So there are a finite number of transitions between congestion states as $q$ varies. ∎

**Proof. (Lemma 2)** By Lemma 1 there are a finite number of transitions between congestion states as $q$ increases. For any interval $(q_i, q_{i+1})$ with $\omega(q) = \bar{\omega}$, $q \in (q_i, q_{i+1})$, we have

$$q = \sum_{j \in \Xi_m(\bar{\omega})} \varepsilon_m + \sum_{k \in \delta^+(\bar{\omega}, m)} \bar{t}_k - \sum_{k \in \delta^-(\bar{\omega}, m)} \bar{t}_k$$

$$- S_{m,-n}(p_m) - \sum_{\ell \in \Xi_m(\bar{\omega}) \backslash \{m\}} S_\ell(p_m)$$

and so $q$ is a strictly decreasing differentiable function of $p_m$ with

$$\frac{dq}{dp_m} = -S'_{m,-n}(p_m) - \sum_{\ell \in \Xi_m(\bar{\omega}) \backslash \{m\}} S'_\ell(p_m), \ q \in (q_1, q_2),$$

thus establishing (4).

It remains to show that $q(p_m)$ is continuous at $\hat{q}$ where $\lim_{\eta \to 0} \omega(\hat{q} - \eta) = \omega_1$ and $\lim_{\eta \to 0} \omega(\hat{q} + \eta) = \omega_2 \neq \omega_1$. This follows from the fact that $p_m$ is strictly increasing and continuous in $q$. A direct argument can be made as follows. At $q = \hat{q}$, $p_m$ has decreased so that $p_m$ equals the price $p_l$ at the other endpoint $l$ of some congested arc $\hat{k}$. For example, assume that $k \in \delta^-(\omega, m)$, so initially $p_m \geq p_l$ and $t_{\hat{k}} = -\bar{t}_{\hat{k}}$. Allowing $p_m < p_l$ would violate the complementary slackness conditions for arc $\hat{k}$, so flow through the arc $\hat{k}$ increases from its lower bound $-\bar{t}_{\hat{k}}$ to become strictly uncongested, while maintaining $p_m = p_l$. We have from the flow balance in the integrated area $\Xi_l(\omega_1)$ that

$$\sum_{j \in \Xi_l(\omega_1)} S_j(p_m) = \sum_{j \in \Xi_l(\omega_1)} \varepsilon_j + \bar{t}_{\hat{k}} + \sum_{r \in \delta^+(\omega_1, l) \backslash \{\hat{k}\}} \bar{t}_r - \sum_{r \in \delta^-(\omega_1, l) \backslash \{\hat{k}\}} \bar{t}_r, \quad (28)$$

Adding (27) with $\omega(q) = \omega_1$ and (28) gives

$$\hat{q} + \sum_{j \in \Xi_m(\omega_1)} S_{j,-n}(p_m) + \sum_{j \in \Xi_l(\omega_1)} S_j(p_m)$$

$$= \sum_{j \in \Xi_m(\omega_1)} \varepsilon_j - \bar{t}_{\hat{k}} + \sum_{k \in \delta^+(\omega_1, m)} \bar{t}_k - \sum_{k \in \delta^-(\omega_1, m) \backslash \{\hat{k}\}} \bar{t}_k$$

$$+ \sum_{j \in \Xi_l(\omega_1)} \varepsilon_j + \bar{t}_{\hat{k}} + \sum_{r \in \delta^+(\omega_1, l) \backslash \{\hat{k}\}} \bar{t}_r - \sum_{r \in \delta^-(\omega_1, l)} \bar{t}_r,$$

$$= \sum_{j \in \Xi_m(\omega_1) \cup \Xi_l(\omega_1)} \varepsilon_j + \sum_{k \in \delta^+(\omega_1, m) \cup \delta^+(\omega_1, l)} \bar{t}_k - \sum_{k \in \delta^-(\omega_1, m) \cup \delta^-(\omega_1, l)} \bar{t}_k$$

$$= \sum_{j \in \Xi_m(\omega_2)} \varepsilon_j + \sum_{k \in \delta^+(\omega_2, m)} \bar{t}_k - \sum_{k \in \delta^-(\omega_2, m)} \bar{t}_k$$

the flow balance in the new congestion state $\omega_2$. We can make a similar argument for $k \in \delta^+(\omega, m)$ and for arcs connecting nodes in the integrated area $\Xi_m(\omega)$ that become congested. ∎

## Optimality conditions

**Proof. (Lemma 3)** For the sake of this argument, assume that firm $m$ makes a Bertrand offer at price $p$ at its node $n$, so that its offer crosses the residual demand curve at price $p$ for any vector of demand shocks. We first show that $P(\{\varepsilon \mid (p, D_{mn}^{\boldsymbol{\varepsilon}}(p)) \text{ is a kink}\}) = 0$. Consider all realizations $\varepsilon$ for which $(p, D_{mn}^{\boldsymbol{\varepsilon}}(p))$ is a kink, where the congestion state changes between $\omega_-$ and $\omega_+$. Then we have one-sided limits for any such realization $\varepsilon$

$$\lim_{r \downarrow p} \omega_{mn}(\boldsymbol{\varepsilon}, r) = \omega_+$$

and

$$\lim_{r \uparrow p} \omega_{mn}(\boldsymbol{\varepsilon}, r) = \omega_-$$

where $\omega_+ \neq \omega_-$. Without loss of generality, we can assume that an arc $k$ with $t_k \in (-\bar{t}_k, \bar{t}_k)$ in $\omega_+$ has $t_k = \bar{t}_k$ in state $\omega_-$. (A similar argument can be applied when $t_k = -\bar{t}_k$ in state $\omega_-$.) Let the endpoints of arc $k$ be denoted node 1 and node 2.

In state $\omega_-$ (where price $r < p$) suppose that the arc $k$ goes from the integrated node set $\Xi_1$ to the integrated node set $\Xi_2$. Let the prices in these sets be $p_1(\boldsymbol{\varepsilon})$ and $p_2(\boldsymbol{\varepsilon})$ where $p_1(\boldsymbol{\varepsilon}) \leq p_2(\boldsymbol{\varepsilon})$. If $m \notin \Xi_1 \cup \Xi_2$ then we have in state $\omega_-$ that

$$\bar{t}_k + \sum_{\ell \in \Xi_1} \varepsilon_\ell = \sum_{\ell \in \Xi_1} S_\ell\left(p_1(\boldsymbol{\varepsilon})\right) + \widetilde{t}_{\Xi_1}$$

where $\widetilde{t}_{\Xi_1}$ is the net flow through congested arcs (not including $k$) into node set $\Xi_1$. Similarly

$$-\bar{t}_k + \sum_{\ell \in \Xi_2} \varepsilon_\ell = \sum_{\ell \in \Xi_2} S_\ell\left(p_2(\boldsymbol{\varepsilon})\right) + \widetilde{t}_{\Xi_2}.$$

The dispatch and nodal prices are continuous as a function of $r$, so in state $\omega_+$ (where price $r > p$) these equations also hold for $p_1(\boldsymbol{\varepsilon}) = p_2(\boldsymbol{\varepsilon}) = p(\boldsymbol{\varepsilon})$. For each $i$, $\sum_{\ell \in \Xi_i} S_\ell(p)$ is an invertible function, so the above means that demand shocks need to satisfy a relation of the form

$$T_1\Big(\sum_{\ell \in \Xi_1} \varepsilon_\ell + \bar{t}_k - \widetilde{t}_{\Xi_1}\Big) = T_2\Big(\sum_{\ell \in \Xi_2} \varepsilon_\ell - \bar{t}_k - \widetilde{t}_{\Xi_2}\Big). \tag{29}$$

Observe that 29 holds for every $\varepsilon$ for which $(p, D_{mn}^{\boldsymbol{\varepsilon}}(p))$ is a kink, where the congestion state changes between $\omega_-$ and $\omega_+$. This restriction removes one degree of freedom and so the set of $\boldsymbol{\varepsilon}$ satisfying it has Lebesgue measure zero. It follows that the probability of this set of demand shocks is zero.

If $m \in \Xi_1$ then $p_1(\boldsymbol{\varepsilon}) = p$. In $\omega_-$ (where price $r < p$) continuity of dispatch and nodal prices gives

$$-\bar{t}_k + \sum_{\ell \in \Xi_2} \varepsilon_\ell \geq \sum_{\ell \in \Xi_2} S_\ell\left(p_2(\boldsymbol{\varepsilon})\right) + \widetilde{t}_{\Xi_2}$$

and in state $\omega_+$ (where price $r > p$) an unconstrained line means

$$-\bar{t}_k + \sum_{\ell \in \Xi_2} \varepsilon_\ell \leq \sum_{\ell \in \Xi_2} S_\ell(p) + \widetilde{t}_{\Xi_2}.$$

Since $S_\ell$ is strictly increasing this means that $p_2(\boldsymbol{\varepsilon}) = p$, and so

$$-\bar{t}_k + \sum_{\ell \in \Xi_2} \varepsilon_\ell = \sum_{\ell \in \Xi_2} S_\ell(p) + \widetilde{t}_{\Xi_2}.$$

Since $p$ is fixed, this restriction removes one degree of freedom and so the set of $\boldsymbol{\varepsilon}$ satisfying it has Lebesgue measure zero, and hence also probability zero.

If $m \in \Xi_2$ then $p_2(\boldsymbol{\varepsilon}) = p$. In state $\omega_-$ a constrained line means

$$\bar{t}_k + \sum_{\ell \in \Xi_1} \varepsilon_\ell \leq \sum_{\ell \in \Xi_1} S_\ell(p_1(\boldsymbol{\varepsilon})) + \widetilde{t}_{\Xi_1}$$

and in state $\omega_1$ an unconstrained line means

$$\bar{t}_k + \sum_{\ell \in \Xi_1} \varepsilon_\ell \geq \sum_{\ell \in \Xi_1} S_\ell(p) + \widetilde{t}_{\Xi_1}.$$

Since $S_\ell$ is strictly increasing this means that $p_1(\boldsymbol{\varepsilon}) = p$, and so

$$\bar{t}_k + \sum_{\ell \in \Xi_1} \varepsilon_\ell = \sum_{\ell \in \Xi_1} S_\ell(p) + \widetilde{t}_{\Xi_1}.$$

Since $p$ is fixed this restriction removes one degree of freedom and so the set of $\boldsymbol{\varepsilon}$ satisfying it has Lebesgue measure zero, and so has probability zero when evaluated with a density function.

The above argument establishes that the set of $\boldsymbol{\varepsilon}$ for which $(p, D_{mn}^{\boldsymbol{\varepsilon}}(p))$ is a point that transitions from one particular congestion state to an adjacent one has probability zero. Taking the union over the finite number of possible transitions shows that $\mathbb{P}(\{\boldsymbol{\varepsilon} \mid (p, D_{mn}^{\boldsymbol{\varepsilon}}(p)) \text{ is a kink}\}) = 0$. Since $\partial E_{mn}(p, q, \omega) \subseteq \mathbb{P}(\{\boldsymbol{\varepsilon} \mid (p, D_{mn}^{\boldsymbol{\varepsilon}}(p)) \text{ is a kink}\})$, we can deduce also that $\mathbb{P}(\partial E_{mn}(p, q, \omega)) = 0$. ∎

**Lemma 7** $\dfrac{\left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})}}{\left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})}} = -D'_{mn}(\hat{p}, \omega).$

**Proof.** Consider a particular state $\omega$. For this state, we have from (27) that $\sum_{j \in \Xi_m(\omega(q))} \varepsilon_j$ is simply an additive demand shock that shifts the ex-post residual demand function horizontally. Thus for a given additive shock $\boldsymbol{\varepsilon}$ corresponding to congestion state $\omega$, $\psi_{mn}(p, D_{mn}^{\boldsymbol{\varepsilon}}(p), \omega) = g(\boldsymbol{\varepsilon})$ along such a curve for some function $g$. We know from Lemma 2 that the slope $D'_{mn}(\hat{p}, \omega)$ of the residual demand curve at $\hat{p}$, is the same for all demand shocks that result in the state $\omega$. Implicit differentiation of $\psi_{mn}(p, D_{mn}^{\boldsymbol{\varepsilon}}(p), \omega) = g(\boldsymbol{\varepsilon})$ with respect to $p$ gives

$$\left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})} + \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})} D'_{mn}(\hat{p}, \omega) = 0$$

from which the result follows. ∎

**Lemma 8** $\sum_{\omega} D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \mid \hat{p}, \hat{q}) = -\dfrac{\left[\frac{\partial \psi_{mn}(p,q)}{\partial p}\right]_{(\hat{p},\hat{q})}}{\left[\frac{\partial \psi_{mn}(p,q)}{\partial q}\right]_{(\hat{p},\hat{q})}}.$

**Proof.** Using (5) and Lemma 7 gives

$$\sum_{\omega} D'_{mn}(p,\omega)\hat{P}_{mn}(\omega \quad \mid \quad \hat{p}, \hat{q})$$

$$= \frac{\sum_{\omega} \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})} D'_{mn}(p,\omega)}{\sum_{\omega} \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})}}$$

$$= -\frac{\sum_{\omega} \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})}}{\sum_{\omega} \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})}}. \tag{30}$$

We have

$$\sum_{\omega} \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial q}\right]_{(\hat{p},\hat{q})} = \left[\frac{\partial \psi_{mn}(p,q)}{\partial q}\right]_{(\hat{p},\hat{q})}$$

and similarly

$$\sum_{\omega} \left[\frac{\partial \psi_{mn}(p,q,\omega)}{\partial p}\right]_{(\hat{p},\hat{q})} = \left[\frac{\partial \psi_{mn}(p,q)}{\partial p}\right]_{(\hat{p},\hat{q})},$$

which yields the result when substituted in (30). ∎

**Proof. (Proposition 1)** As shown by Anderson and Philpott (2002a) a monotonic increasing supply function $Q_{mn}(p)$ is globally optimal if it satisfies:

$$\begin{cases} \tilde{Z}(p,q) \geq 0 & \text{if } q < Q_{mn}(p) \\ \tilde{Z}(p,q) = 0 & \text{if } q = Q_{mn}(p) \\ \tilde{Z}(p,q) \leq 0 & \text{if } q > Q_{mn}(p). \end{cases}$$

where

$$\tilde{Z}(p,q) = (p - C'_{mn}(q))\frac{\partial \psi_{mn}}{\partial p} - q\frac{\partial \psi_{mn}}{\partial q}.$$

If $\frac{\partial \psi_{mn}}{\partial q} > 0$ then we have from Lemma 8 and Definition 2 that

$$\begin{aligned} Z(p,q) &= \tilde{Z}(p,q)/\frac{\partial \psi_{mn}}{\partial q} \\ &= (p - C'_{mn}(q))\sum_{\omega} -D'_{mn}(p,\omega))\hat{P}(\omega \mid p,q) - q \\ &= (p - C'_{mn}(q))\left(\frac{\partial \psi_{mn}}{\partial p}/\frac{\partial \psi_{mn}}{\partial q}\right) - q. \end{aligned}$$

Since

$$\tilde{Z}(p,q \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0 \iff Z(p,q) \begin{Bmatrix} > \\ = \\ < \end{Bmatrix} 0$$

we have that a supply function $Q_{mn}(p)$ is globally optimal if it satisfies

$$\begin{cases} Z_{mn}(p,q) \geq 0 & \text{if } q < Q_{mn}(p) \\ Z_{mn}(p,q) = 0 & \text{if } q = Q_{mn}(p) \\ Z_{mn}(p,q) \leq 0 & \text{if } q > Q_{mn}(p). \end{cases}$$

as required. ∎

**Proof. (Lemma 5)**We use (4) with $q = Q(p)$, $S_\ell(p) = NQ(p)$ and $S_{m,-n}(p) = (N-1)Q(p)$ to determine $D'_{mn}(p,\omega)$. Next we substitute $D'_{mn}(p,\omega)$ into (6), and then use the first-order condition $Z_{mn}(p,q) = 0$.

$$Q = (p - C'(Q)) \sum_\omega \left( (N-1)Q' + \sum_{\ell \in \Xi_m(\omega)\setminus\{m\}} NQ' \right) \hat{P}_{mn}(\omega|p,Q).$$

We have $\sum_\omega \hat{P}_{mn}(\omega|p,q) = 1$, $M_{\Xi_m(\omega)} = \sum_{\ell \in \Xi_m(\omega)} 1$, and by definition $m \in \Xi_m(\omega)$, so it follows from Definition 3 that

$$\sum_\omega \sum_{\ell \in \Xi_m(\omega)\setminus\{m\}} \hat{P}_{mn}(\omega|p,q) = \mu_{mn}(p,Q) - 1.$$

Thus

$$Q = (p - C'(Q))((N-1)Q' + (\mu_{mn}(p,Q) - 1)NQ'),$$

which gives

$$Q = (p - C'(Q))(\mu_{mn}(p,Q(p))N - 1)Q'.$$


Next, we show that $\widetilde{\mu}(q)$ is independent of the choice of $T_m(q)$. Assume that $\mathbf{S}(\mathbf{p})$ is a vector of identical nodal supply functions $s(p)$. Assume that all nodal supply functions change to $\hat{s}(p)$. Such a change would typically change nodal prices. But for a given realized vector of demand shocks $\boldsymbol{\varepsilon}$, we conjecture that the change will not change cleared nodal production, network flows or the congestion state. We prove that the conjecture is correct by verifying that it satisfies the market-clearing conditions in (3) for any vector of shock outcomes. First we note that conjectured nodal output and flows will satisfy (2). Next, the new (conjectured) nodal price $\hat{p}_\ell$ for a node $\ell$ can be calculated for each shock outcome from the old nodal price $p_\ell$ and the conjecture that $\hat{s}(\hat{p}_\ell) = s(p_\ell)$. New supply functions are monotonic and identical in each node and it has been conjectured that nodal output is unchanged. Thus for a given shock outcome, new prices $\widehat{\mathbf{p}}$ are such that

$$\widehat{p}_m \geq \widehat{p}_\ell \text{ if and only if } p_m \geq p_\ell. \tag{31}$$

New shadow prices can be calculated for each arc from the first market-clearing condition in (3) by calculating price differences between nodes that are connected by the arc. It follows from (31) that the new shadow prices will have the same sign as the old shadow prices. Thus the new shadow prices will satisfy the complementary slackness conditions in (3), because flows are the same and the old shadow

prices satisfied those conditions, which verify our conjecture. The argument proves that the considered change of nodal supply functions does not change the congestion state for any vector of local demand shocks. Thus $\widetilde{\mu}(q)$ is independent of the choice of $T_m(q)$. ∎

## Computing $\psi_{mn}(p, q, \omega)$

Any incidence matrix $\mathbf{A}$ has rank $M - 1$. However, we can remove any row from the matrix $\mathbf{A}$ to make it have full rank, and therefore be invertible in the radial case (Bapat, 2010). The removed row corresponds to a node, which we denote by $m$ and refer to as the *slack node* (or *swing bus* in power systems terminology). We can now write the market-clearing condition that net-imports equal net-demand in the remaining nodes as follows

$$\mathbf{A}_{-m}\mathbf{t} = \boldsymbol{\varepsilon}_{-m} - \mathbf{s}_{-m}(\mathbf{p}),$$

where we use the subscript $-m$ to indicate that row $m$ has been removed. Flows in the network can now be determined from net-exports in the remaining nodes as follows:

$$\mathbf{t} = \underbrace{-\left(\mathbf{A}_{-m}\right)^{-1}}_{\mathbf{H}}\left(\mathbf{s}_{-m}(\mathbf{p}) - \boldsymbol{\varepsilon}_{-m}\right). \tag{32}$$

In general the components of the matrix $\mathbf{H}$ define power transfer distribution factors (PTDFs) $H_{k\ell}$ that give the flow on arc $k$ that would result from a net-injection of one unit at node $\ell \neq m$ and a withdrawal of one unit at the slack node $m$.

Similar to Wilson (2008) it is convenient to choose the slack node $m$ to be a trading hub with nodal price $p = p_m$. As shown by Xu and Baldick (2007), Hogan (2000) and Chao et al. (2000), one can express the vector of other nodal prices $\mathbf{p}_{-m}$ in terms of the price of the trading hub and the shadow prices of the arcs.

$$\mathbf{p}_{-m} = p\mathbf{1}_{M-1} - \mathbf{H}^T\left(\boldsymbol{\rho} - \boldsymbol{\sigma}\right), \tag{33}$$

where $\mathbf{1}_{M-1}$ is a column vector of $M - 1$ ones. Thus the injection of one unit at node $\ell \neq m$ that is withdrawn at the trading hub $m$ is paid $p$ (the local price at the trading hub) minus resulting shadow price payments for resulting flows on congested lines.

The proofs below also use the following notation. $M_{\Xi_m(\omega)}$ is the number of nodes in $\Xi_m(\omega)$ and we note that they must be connected by $M_{\Xi_m(\omega)} - 1$ uncongested arcs. The set $F(\omega)$ contains all other nodes in the network. Similarly we partition the shock vector into $\varepsilon_{\Xi_m(\omega)}$ and $\varepsilon_{F(\omega)}$. Let $\kappa(\omega)$ be the set of uncongested arcs that connect nodes in $\Xi_m(\omega)$. Other arcs are in the set $\vartheta(\omega)$. We let $t_{\kappa(\omega)}$ be the flows in the uncongested arcs between nodes in the set $\Xi_m(\omega)$ and we let $t_{\vartheta(\omega)}$ be the vector of flows in the other arcs. The node-arc incidence matrix $\mathbf{A}_{\Xi_m(\omega)}$ describes the subtree with nodes in $\Xi_m(\omega)$ that are connected by arcs in $\kappa(\omega)$. We let $\mathbf{A}_{\backslash\Xi_m(\omega)}$ be a node-arc incidence matrix with $M - M_{\Xi_m(\omega)}$ rows and $M - M_{\Xi_m(\omega)}$ columns, describing the rest of the network.[1]

---

[1] Note that the remainder of the network has at least one arc that is lacking its start or end node. Also the remainder of the network is not necessarily connected.

**Proof.** (**Lemma 4**) The proof shows how to transform the volume in $\boldsymbol{\varepsilon}-$space into a corresponding volume in $\mathbf{t}$, $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ space for variables that are not at a bound. To do this it suffices to show that $\left| \frac{\partial \boldsymbol{\varepsilon}}{\partial (\mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_{\ell})} \right|$ the absolute value of the determinant of the Jacobian matrix representing the change in measure (see Apostol, 1974) is equal to

$$J(\omega) = \left| \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial \left( \left( \mathbf{t}_{\vartheta(\omega)} \right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)} \right)} \right|.$$

The node $\ell$ can be chosen freely, but in the following we will choose it such that $\ell = m$. It can be shown from the identity $\boldsymbol{\varepsilon} = \mathbf{At} + \mathbf{s(p)}$ that:

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial \left( \mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_m \right)} = \begin{bmatrix} \frac{\partial \boldsymbol{\varepsilon}_{\Xi m(\omega)}}{\partial \left( \mathbf{t}_{\kappa(\omega)} \right)} & \frac{\partial \boldsymbol{\varepsilon}_{\Xi m(\omega)}}{\partial \left( \left( \mathbf{t}_{\vartheta(\omega)} \right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)} \right)} & \frac{\partial \boldsymbol{\varepsilon}_{\Xi m(\omega)}}{\partial \varepsilon_m} \\ \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial \left( \mathbf{t}_{\kappa(\omega)} \right)} & \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left( \left( \mathbf{t}_{\vartheta(\omega)} \right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)} \right)} & \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial \varepsilon_m} \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{A}_{\Xi m(\omega)} & \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}_{\Xi m(\omega)}}{\partial \varepsilon_m} \\ \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left( \left( \mathbf{t}_{\vartheta(\omega)} \right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)} \right)} & \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial \varepsilon_m} \end{bmatrix} \quad (34)$$

Let

$$\mathbf{B} = \begin{bmatrix} \mathbf{A}_{\Xi m(\omega)} & \mathbf{0} \\ \mathbf{0} & \frac{\partial \boldsymbol{\varepsilon}_F}{\partial \left( \left( \mathbf{t}_{\vartheta(\omega)} \right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)} \right)} \end{bmatrix}. \quad (35)$$

We can expand the determinant $\left| \frac{\partial \boldsymbol{\varepsilon}}{\partial \left( \mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_{\ell} \right)} \right|$ in (34) along its $M$th column $\begin{bmatrix} \frac{\partial \boldsymbol{\varepsilon}_{\Xi m(\omega)}}{\partial \varepsilon_m} \\ \frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial \varepsilon_m} \end{bmatrix}$, which has a one in row $m$ and zeros in the other rows, so it follows from the definition of the determinant that:

$$\frac{\partial \boldsymbol{\varepsilon}}{\partial \left( \mathbf{t}_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)}, \varepsilon_m \right)} = \left| (-1)^{m+M} \det \left( \mathbf{B}_{-m} \right) \right| = \left| \det \left( \mathbf{B}_{-m} \right) \right|$$

$$= \left| \left( \mathbf{A}_{\Xi m(\omega)} \right)_{-m} \right| J(\omega),$$

because $\mathbf{B}_{-m}$ is a block matrix with determinant $\left| \left( \mathbf{A}_{\Xi m(\omega)} \right)_{-m} \right| J(\omega)$. $\mathbf{A}_{\Xi m(\omega)}$ is a node-arc incidence matrix of a connected radial network. Thus it follows from Bapat (2010, p. 13) that $\left| \det \left( \mathbf{A}_{\Xi m(\omega)} \right)_{-m} \right|$ is 1, which gives the stated result. ∎

**Lemma 9** *The Jacobian matrix* $\frac{\partial \boldsymbol{\varepsilon}_{F(\omega)}}{\partial \left( \left( \mathbf{t}_{\vartheta(\omega)} \right)_{B(\omega)}, \boldsymbol{\rho}_{U(\omega)}, \boldsymbol{\sigma}_{L(\omega)} \right)}$ *can be constructed for the*

*state $\omega$ from the following results for nodes $\ell \in F(\omega)$:*

$$\frac{\partial \varepsilon_\ell}{\partial \rho_k} = -S'_\ell(p_\ell) \mathbf{H}_{k\ell} \ \text{for } k \in U(\omega)$$

$$\frac{\partial \varepsilon_\ell}{\partial \sigma_k} = S'_\ell(p_\ell) \mathbf{H}_{k\ell} \ \text{for } k \in L(\omega)$$

$$\frac{\partial \varepsilon_\ell}{\partial t_k} = A_{\ell k} \ \text{for } k \in B(\omega).$$

**Proof.** We partition the columns of $\mathbf{A}_{\backslash \Xi_m(\omega)}$ into $\left(\mathbf{A}_{\backslash \Xi_m(\omega)}\right)_{L(\omega)}$, $\left(\mathbf{A}_{\backslash \Xi_m(\omega)}\right)_{B(\omega)}$ and $\left(\mathbf{A}_{\backslash \Xi_m(\omega)}\right)_{U(\omega)}$, corresponding to flows $\mathbf{t}_\vartheta$ being at their lower bounds, strictly between their bounds, and at their upper bounds. Thus the flow balance in (2) can be written as follows

$$\left(\mathbf{A}_{\backslash \Xi_m}\right)_B (\mathbf{t}_\vartheta)_B + \left(\mathbf{A}_{\backslash \Xi_m}\right)_U (\mathbf{t}_\vartheta)_U + \left(\mathbf{A}_{\backslash \Xi_m}\right)_L (\mathbf{t}_\vartheta)_L + \mathbf{s}_F(\mathbf{p}) = \varepsilon_F. \quad (36)$$

Observe that (33) implies that

$$\frac{\partial \varepsilon_\ell}{\partial \rho_k} = \frac{\partial \varepsilon_\ell}{\partial p_\ell} \frac{\partial p_\ell}{\partial \rho_k} = -S'_\ell(p_\ell) \mathbf{H}_{k\ell} \ \text{for } \ell \in F(\omega) \ \text{and } k \in U(\omega)$$

and

$$\frac{\partial \varepsilon_\ell}{\partial \sigma_k} = \frac{\partial \varepsilon_\ell}{\partial p_\ell} \frac{\partial p_\ell}{\partial \sigma_k} = S'_\ell(p_\ell) \mathbf{H}_{k\ell} \ \text{for } \ell \in F(\omega) \ \text{and } k \in L(\omega).$$

Moreover,

$$\frac{\partial \varepsilon_\ell}{\partial t_k} = A_{\ell k} \ \text{for } \ell \in F(\omega) \ \text{and } k \in B(\omega),$$

which gives the result. $\blacksquare$

# Appendix B: Proofs for Section 3

**Proof. (Lemma 6)**. Below we list the congestion states of the network and how we partition the nodes for each state:

| State | $t$ | $\rho$ | $\sigma$ | $\Xi$ | $F(\omega)$ |
|---|---|---|---|---|---|
| $\omega_1$ | $\in (-\bar{t}, \bar{t})$ | 0 | 0 | $\{1, 2\}$ | $\emptyset$ |
| $\omega_2$ | $\bar{t}$ | $\in (0, \infty)$ | 0 | $\{1\}$ | $\{2\}$ |
| $\omega_3$ | $-\bar{t}$ | 0 | $\in (0, \infty)$ | $\{1\}$ | $\{2\}$ |

We have from (2) that

$$\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \end{bmatrix} = \underbrace{\begin{bmatrix} S_1(p_1) \\ S_2(p_2) \end{bmatrix}}_{\mathbf{s}(\mathbf{p})} + \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\mathbf{A}} t \quad (37)$$

and from (32) that

$$\mathbf{H} = -\left(\mathbf{A}_{-1}\right)^{-1} = -1. \quad (38)$$

Thus it follows from (33) that

$$p_2 = p + \rho - \sigma. \qquad (39)$$

The network is completely integrated in state $\omega_1$, so $\boldsymbol{\varepsilon}_{F(\omega_1)}$ is empty. We only need the substitution factor $J(\omega)$ for states $\omega_2$ and $\omega_3$. It follows from Lemma 9, (13) and (38) that

$$J(\omega_2) = \left|\frac{\partial \varepsilon_2}{\partial \rho}\right| = S_2'(p_2) = S_2'(p + \rho)$$
$$J(\omega_3) = \left|\frac{\partial \varepsilon_2}{\partial \sigma}\right| = |-S_2'(p_2)| = S_2'(p - \sigma).$$

(12) and (37) now yields:

$$\frac{\partial \psi_{mn}(p, q, \omega_1)}{\partial q} = \int_{-\bar{t}}^{\bar{t}} f\left(\mathbf{A}t + \mathbf{s}\left(\mathbf{p}, q\right)\right) dt = \int_{-\bar{t}}^{\bar{t}} f\left(q + S_{1,-n}(p) - t, S_2(p) + t\right) dt,$$

$$\frac{\partial \psi_{mn}(p, q, \omega_2)}{\partial q} = \int_0^\infty f\left(\mathbf{A}t + \mathbf{s}\left(\mathbf{p}, q\right)\right) J(\omega_2) d\rho$$
$$= \int_0^\infty f\left(q + S_{1,-n}(p) - \bar{t}, S_2(p + \rho) + \bar{t}\right) S_2'(p + \rho) d\rho$$

and

$$\frac{\partial \psi_{mn}(p, q, \omega_3)}{\partial q} = \int_0^\infty f\left(\mathbf{A}t + \mathbf{s}\left(\mathbf{p}, q\right)\right) J(\omega_3) d\sigma$$
$$= \int_0^\infty f\left(q + S_{1,-n}(p) + \bar{t}, S_2(p - \sigma) - \bar{t}\right) S_2'(p - \sigma) d\sigma.$$

This gives us (16) after the substitutions $\varepsilon_2 = S_2(p + \rho) + \bar{t}$ and $\varepsilon_2 = S_2(p - \sigma) - \bar{t}$, respectively, have been applied to the integrals of the states $\omega_2$ and $\omega_3$. The equation (15) follows from (4), Definition 2 and that the two nodes are only completely integrated in state $\omega_1$. ∎

**Proof. (Proposition 2).** Symmetry of the network, costs and shock densities ensure that the optimal supply functions of all producers are given by identical optimality conditions. We have $S_2(p) = q + S_{1,-n}(p) = NQ(p)$ in a symmetric equilibrium with inelastic demand, so (19) follows from (16). The differential equation in the statement follows from Lemma 5. The function $h(q, \omega_i)$ gives the probability of being in state $\omega_i$ when the total nodal output in a node $m$ is $q$. Thus $\widetilde{\mu}(q) = 1 + \frac{h(q, \omega_1)}{\sum_\omega h(q, \omega)}$. In case that production capacity would bind at some price $p_b < \bar{p}$ then $Q(p)$ is inelastic in the range $(p_b, \bar{p})$, and it follows from (15) that $Z(p, q) < 0$ when $q \in (0, \bar{q})$ and $p \in (p_b, \bar{p})$. This would violate the second-order condition in (8), and it is necessary that this condition is locally satisfied (Anderson and Philpott, 2002a). Thus the production capacity must bind at the reservation price, which gives our initial condition.

Next we show that the symmetric solution is unique. It follows from the assumptions for $f(\varepsilon_1, \varepsilon_2)$, our definition of $h(NQ, \omega)$ and from (18) that

$$\frac{1}{(N\widetilde{\mu}(NQ) - 1)} > 0,$$

and that $\frac{1}{(N\widetilde{\mu}(NQ) - 1)}$ is Lipschitz continuous in $Q$. Consider a price $\tilde{p} \in (C'(0), \bar{p})$. We now want to show that $p - C'(Q(p))$ is bounded away from zero in the range $[\tilde{p}, \bar{p}]$. This is obvious for constant marginal costs, as we then have that $\tilde{p} - C'(Q(\tilde{p})) = \tilde{p} - C'(0) > 0$. For strictly increasing marginal costs we can use the

following argument. It follows from Picard-Lindelöf's theorem and $\overline{p} > C'(\overline{q})$ that a unique solution to (17) must exist for some range $[p_0, \overline{p}]$. In this price range the mark-up, $p - C'(Q(p))$, is smallest at some price $p^*$ where the inverse supply function is at least as steep as the marginal cost curve, i.e. $Q'(p^*) \leq \frac{1}{C''(Q(p^*))}$. Thus it follows from (17) that

$$p^* - C'(Q(p^*)) \geq \frac{Q(p^*)C''(Q(p^*))}{(N\tilde{\mu}(NQ(p^*))-1)}.$$

This is bounded away from zero whenever $Q(p^*)$ is bounded away from zero if marginal costs are strictly increasing. In case $Q(p^*) = 0$ for some price $p^* > C'(0)$, it follows from (17) that $Q'(p) = 0$ for $p \in (\tilde{p}, p^*)$. Thus it follows from Picard-Lindelöf's theorem and the properties of (17) that a unique monotonic symmetric solution will exist for the price interval $[\tilde{p}, \overline{p}]$. We can repeat the argument for any $\tilde{p} \in (C'(0), \overline{p})$ to show that a unique monotonic symmetric solution will exist for the price interval $(C'(0), \overline{p}]$.

We now verify the global second-order conditions. To somewhat simplify notation let

$$P(p, q, \omega) = \frac{\partial \psi_{mn}(p, q, \omega)}{\partial q},$$

$$\beta(p, q) = (2N - 1) P(p, q, \omega_1) + (N - 1) P(p, q, \omega_2) + (N - 1) P(p, q, \omega_3), \quad (40)$$

and

$$P(p, q) = P(p, q, \omega_1) + P(p, q, \omega_2) + P(p, q, \omega_3). \quad (41)$$

We have from (5) and (15) that

$$Z(p, q) = \frac{(p - C'(q))\beta(p, q)Q'(p)}{P(p, q)} - q.$$

We also have $C'' \geq 0$ and $Q'(p) \geq 0$, so

$$Z_q \leq \frac{(p - C'(q))\beta_q Q' P - (p - C'(q))\beta Q' P_q}{(P(p, q))^2} - 1.$$

In particular, whenever $Z(p, q) = 0$, we have

$$Z_q \leq \frac{qP\beta_q - \beta P - q\beta P_q}{\beta P}.$$

We know from (8) that the solution is an equilibrium if $Z(p, q) \geq 0$ when $q \leq Q(p)$ and $Z(p, q) \leq 0$ when $q \geq Q(p)$. This follows if $Z_q(p, q) \leq 0$ whenever $Z(p, q) = 0$. To verify this sufficiency condition, it suffices to show that

$$\beta(p, q)P(p, q) + q\beta(p, q)P_q(p, q) - qP(p, q)\beta_q(p, q) \geq 0. \quad (42)$$

To show this observe that the assumption

$$2N\overline{q} |f_m(\varepsilon_1, \varepsilon_2)| \leq (3N - 2) f(\varepsilon_1, \varepsilon_2)$$

implies from (16) that

$$
\begin{aligned}
2Nq\left|P_q\left(p,q,\omega_1\right)\right| &= 2Nq\left|\int_{-\bar{t}}^{\bar{t}}\frac{\partial}{\partial q}f\left(q+S_{1,-n}\left(p\right)-t,S_2\left(p\right)+t\right)dt\right|\\
&\leq 2N\bar{q}\left|\int_{-\bar{t}}^{\bar{t}}\frac{\partial}{\partial q}f\left(q+S_{1,-n}\left(p\right)-t,S_2\left(p\right)+t\right)dt\right|\\
&\leq \int_{-\bar{t}}^{\bar{t}}2N\bar{q}\left|f_1\left(q+S_{1,-n}\left(p\right)-t,S_2\left(p\right)+t\right)\right|dt\\
&\leq (3N-2)\int_{-\bar{t}}^{\bar{t}}f\left(q+S_{1,-n}\left(p\right)-t,S_2\left(p\right)+t\right)dt\\
&= (3N-2)P\left(p,q,\omega_1\right).
\end{aligned}
$$

Similarly $2Nq\left|P_q\left(p,q,\omega_3\right)\right|\leq(3N-2)P\left(p,q,\omega_3\right)$ and $2Nq\left|P_q\left(p,q,\omega_2\right)\right|\leq(3N-2)P\left(p,q,\omega_2\right)$. It follows from (41) and (40) that

$$
\begin{aligned}
&q\beta(p,q)P_q(p,q)-qP(p,q)\beta_q(p,q)\\
=~&qN\left(P\left(p,q,\omega_1\right)\left(P_q(p,q,\omega_2)+P_q\left(p,q,\omega_3\right)\right)-qNP_q\left(p,q,\omega_1\right)\left(P(p,q,\omega_2)+P\left(p,q,\omega_3\right)\right)\right)\\
\geq~&-(3N-2)P\left(p,q,\omega_1\right)\left(P(p,q,\omega_2)+P\left(p,q,\omega_3\right)\right).
\end{aligned}
$$

It can be deduced from (41) and (40) that

$$
\beta(p,q)P(p,q)\geq(3N-2)P\left(p,q,\omega_1\right)\left(P(p,q,\omega_2)+P\left(p,q,\omega_3\right)\right).
$$

Thus (42) is satisfied, which is sufficient for an equilibrium. ■

**Proof. (Proposition 3)** Under Assumption 1, we have from (16) that:

$$
\begin{aligned}
\frac{\partial\psi_{mn}(p,q,\omega_1)}{\partial q} &= \int_{-\bar{t}}^{\bar{t}}f\left(q+S_{1,-n}\left(p\right)-t,S_2\left(p\right)+t\right)dt = \int_{-\bar{t}}^{\bar{t}}\frac{dt}{V_1} = \frac{2\bar{t}}{V_1}\\
\frac{\partial\psi_{mn}(p,q,\omega_2)}{\partial q} &= \int_{S_2(p)+\bar{t}}^{\infty}f\left(q+S_{1,-n}\left(p\right)-\bar{t},\varepsilon_2\right)d\varepsilon_2 = \int_{S_2(p)+\bar{t}}^{N\bar{q}+\bar{t}}\frac{d\varepsilon_2}{V_1} = \frac{N\bar{q}-S_2(p)}{V_1} \qquad(43)\\
\frac{\partial\psi_{mn}(p,q,\omega_3)}{\partial q} &= \int_{-\infty}^{S_2(p)-\bar{t}}f\left(q+S_{1,-n}\left(p\right)+\bar{t},\varepsilon_2\right)d\varepsilon_2 = \int_{-\bar{t}}^{S_2(p)-\bar{t}}\frac{d\varepsilon_2}{V_1} = \frac{S_2(p)}{V_1}.
\end{aligned}
$$

(20) now follows from (18). For constant $\mu$, we note the similarities between (14) and the first-order condition for single-node networks with $\widetilde{N}$ symmetric firms (Klemperer and Meyer, 1989).

$$
Q = (p - C'(Q))\left(\widetilde{N}-1\right)Q'. \qquad(44)
$$

By comparing (14) and (44) we can conclude that the first-order solution of a firm in a symmetric two-node network with $N$ firms per node is the same as for a firm in an isolated node with inelastic demand and $\mu N$ symmetric firms. Thus analytical solutions to (44), derived by Anderson and Philpott (2002b) and Rudkevich et al. (1998), are also solutions to (14) when $\widetilde{N}=\mu N$, which gives us (21). We also

know that such solutions are monotonic (Holmberg, 2008). It follows from our assumptions and Proposition 2 that this is a supply-function equilibrium. ∎

**Proof. (Proposition 4)** Local net-imports must equal net-demand in every node, so

$$
\begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{bmatrix} = \underbrace{\begin{bmatrix} S_1(p_1) \\ S_2(p_2) \\ S_3(p_3) \\ S_4(p_4) \end{bmatrix}}_{\mathbf{s(p)}} + \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} t_1(\mathbf{p}) \\ t_2(\mathbf{p}) \\ t_3(\mathbf{p}) \end{bmatrix}}_{\mathbf{t}}. \tag{45}
$$

We derive the optimal supply function for a producer in node 1, so we choose this node to be the slack node and trading hub. Thus

$$
\mathbf{A}_{-1} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 1 & 1 \end{bmatrix} \tag{46}
$$

and we have from (32) that

$$
\mathbf{H} = -(\mathbf{A}_{-1})^{-1} = \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \tag{47}
$$

so it follows from (33) that

$$
\mathbf{p}_{-1} = p \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} (\boldsymbol{\rho} - \boldsymbol{\sigma}). \tag{48}
$$

Each arc $k$ has three states. In the uncongested state we have $\sigma_k = 0$, $\rho_k = 0$ and $t_k \in (-\bar{t}, \bar{t})$. When the arc is congested towards node 4 we have $t_k = \bar{t}$, $\sigma_k = 0$, and $\rho_k > 0$ and when the arc is congested away from node 4 we have $t_k = -\bar{t}$, $\sigma_k > 0$, and $\rho_k = 0$. Altogether there are $3 \times 3 \times 3 = 27$ congestion states. In Holmberg and Philpott (2015), we use (12) to calculate $\partial \psi(p, q, \omega)/\partial q$ for one state $\omega$ at a time. The results are summarized in Table 1. Each competitor is assumed to submit a symmetric offer $Q(p)$, so $S_2(p) \equiv S_3(p) \equiv S(p) := NQ(p)$. Adding the results in Table 1 yields:

$$
\sum_{\omega} \frac{\partial \psi(p, q, \omega)}{\partial q} = \frac{6\bar{t}S^2(\bar{p})}{V} + \frac{16\bar{t}^2 S(\bar{p})}{V} + \frac{8\bar{t}^3}{V}. \tag{49}
$$

Node 1 is completely integrated with either node 2 or 3 in states $\omega_{15}$, $\omega_{17}$, $\omega_{26}$, $\omega_{27}$ and completely integrated with both nodes in state $\omega_{18}$. In the other states node 1 is either isolated or only completely integrated with node 4, which does not have any producers and where demand is inelastic. We have

$$
\begin{aligned}
&\frac{\partial \psi(p,q,\omega_{15})}{\partial q} + \frac{\partial \psi(p,q,\omega_{17})}{\partial q} + \frac{\partial \psi(p,q,\omega_{26})}{\partial q} + \frac{\partial \psi(p,q,\omega_{27})}{\partial q} + 2\frac{\partial \psi(p,q,\omega_{18})}{\partial q} \\
&= \frac{4\bar{t}^2 S(p)}{V} + \frac{4\bar{t}^2(S(\bar{p})-S(p))}{V} + \frac{4\bar{t}^2 S(p)}{V} + \frac{4\bar{t}^2(S(\bar{p})-S(p))}{V} + \frac{16\bar{t}^3}{V} = \frac{8\bar{t}^2 S(\bar{p})+16\bar{t}^3}{V},
\end{aligned} \tag{50}
$$

Table 1: The 27 congestion states of the star network.

| State | $t_1(\omega)$ | $t_2(\omega)$ | $t_3(\omega)$ | $\frac{\partial\psi(p,q,\omega)}{\partial q}$ |
|---|---|---|---|---|
| $\omega_1$ | $\bar{t}$ | $\bar{t}$ | $\bar{t}$ | $0$ |
| $\omega_2$ | $\bar{t}$ | $\bar{t}$ | $-\bar{t}$ | $0$ |
| $\omega_3$ | $\bar{t}$ | $\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{\bar{t}\left(S^2(\bar{p})-S^2(p)\right)}{V}$ |
| $\omega_4$ | $\bar{t}$ | $-\bar{t}$ | $-\bar{t}$ | $0$ |
| $\omega_5$ | $\bar{t}$ | $-\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{\bar{t}(S(\bar{p})-S(p))^2}{V}$ |
| $\omega_6$ | $\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{8\bar{t}^2(S(\bar{p})-S(p))}{V}$ |
| $\omega_7$ | $-\bar{t}$ | $\bar{t}$ | $\bar{t}$ | $0$ |
| $\omega_8$ | $-\bar{t}$ | $\bar{t}$ | $-\bar{t}$ | $0$ |
| $\omega_9$ | $-\bar{t}$ | $\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{\bar{t}S^2(p)}{V}$ |
| $\omega_{10}$ | $-\bar{t}$ | $-\bar{t}$ | $-\bar{t}$ | $0$ |
| $\omega_{11}$ | $-\bar{t}$ | $-\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{\bar{t}S(p)(2S(\bar{p})-S(p))}{V}$ |
| $\omega_{12}$ | $-\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{8\bar{t}^2 S(p)}{V}$ |
| $\omega_{13}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\bar{t}$ | $\bar{t}$ | $\frac{2\bar{t}S^2(p)}{V}$ |
| $\omega_{14}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\bar{t}$ | $-\bar{t}$ | $\frac{2\bar{t}S(p)(S(\bar{p})-S(p))}{V}$ |
| $\omega_{15}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{4\bar{t}^2 S(p)}{V}$ |
| $\omega_{16}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $-\bar{t}$ | $-\bar{t}$ | $\frac{2\bar{t}(S(\bar{p})-S(p))^2}{V}$ |
| $\omega_{17}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $-\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{4\bar{t}^2(S(\bar{p})-S(p))}{V}$ |
| $\omega_{18}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\frac{8\bar{t}^3}{V}$ |
| $\omega_{19}$ | $\bar{t}$ | $-\bar{t}$ | $\bar{t}$ | $0$ |
| $\omega_{20}$ | $\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\bar{t}$ | $\frac{\bar{t}\left(S^2(\bar{p})-S^2(p)\right)}{V}$ |
| $\omega_{21}$ | $\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $-\bar{t}$ | $\frac{\bar{t}(S(\bar{p})-S(p))^2}{V}$ |
| $\omega_{22}$ | $-\bar{t}$ | $-\bar{t}$ | $\bar{t}$ | $0$ |
| $\omega_{23}$ | $-\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\bar{t}$ | $\frac{\bar{t}S^2(p)}{V}$ |
| $\omega_{24}$ | $-\bar{t}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $-\bar{t}$ | $\frac{\bar{t}S(p)(2S(\bar{p})-S(p))}{V}$ |
| $\omega_{25}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $-\bar{t}$ | $\bar{t}$ | $\frac{2\bar{t}S(p)(S(\bar{p})-S(p))}{V}$ |
| $\omega_{26}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\bar{t}$ | $\frac{4\bar{t}^2 S(p)}{V}$ |
| $\omega_{27}$ | $\in\left(-\bar{t},\bar{t}\right)$ | $\in\left(-\bar{t},\bar{t}\right)$ | $-\bar{t}$ | $\frac{4\bar{t}^2(S(\bar{p})-S(p))}{V}$ |

and

$$\hat{P}\left(\omega_{15}|p,q\right) + \hat{P}\left(\omega_{17}|p,q\right) + \hat{P}\left(\omega_{26}|p,q\right) + \hat{P}\left(\omega_{27}|p,q\right) + 2\hat{P}\left(\omega_{18}|p,q\right)$$
$$= \frac{4\bar{t}S(\bar{p}) + 8\bar{t}^2}{3S^2(\bar{p}) + 8\bar{t}S(\bar{p}) + 4\bar{t}^2}. \tag{51}$$

This gives (26), because $S\left(\bar{p}\right) := N\bar{q}$, and

$$\mu = 1 + \hat{P}\left(\omega_{15}|p,q\right) + \hat{P}\left(\omega_{17}|p,q\right) + \hat{P}\left(\omega_{26}|p,q\right) + \hat{P}\left(\omega_{27}|p,q\right) + 2\hat{P}\left(\omega_{18}|p,q\right).$$

It follows from (4), Definition 2, (49) and (50) that

$$Z\left(p,q\right) = \frac{\tilde{Z}\left(p,q\right)}{\frac{2\bar{t}}{V}\left[3S^2\left(\bar{p}\right) + 8\bar{t}S\left(\bar{p}\right) + 4\bar{t}^2\right]}$$

where

$$\tilde{Z}\left(p,q\right) = \left(p - C'(q)\right)\left(S'_{1,-i}\left(p\right)\left(\frac{6\bar{t}S^2(\bar{p})}{V} + \frac{16\bar{t}^2S(\bar{p})}{V}\right) + \frac{8\bar{t}^2S(\bar{p}) + 16\bar{t}^3}{V}S'\left(p\right)\right)$$
$$- q\frac{2\bar{t}}{V}\left[3S^2\left(\bar{p}\right) + 8\bar{t}S\left(\bar{p}\right) + 4\bar{t}^2\right].$$

We note that $\frac{\partial Z(p,q)}{\partial q} = \frac{\partial \tilde{Z}(p,q)}{\partial q} \leq 0$, so if we find a monotonic stationary solution, then it is an equilibrium. The explicit equilibrium expression and monotonicity of this solution can be established as in the proof of Proposition 3. ∎