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Modelling Network Constrained Economic Dispatch Problems

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Abstract The behavior of DC load flow formulations when they are used in economic dispatch and nodal pricing models is discussed. It is demonstrated that non-negative prices in these models are sufficient to guarantee global optimality of any local optimum, even if the feasible region is not convex, and so a negative nodal price is an indicator of a possible loss in optimality. We also discuss the possible effect that negative prices might have on algorithms that assume this convexity.

Keywords Convex Optimization · Energy Systems · Economic Dispatch · Quadratic Programming · Linear Programming

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1 Introduction

In power markets there is an increasing need for improving the representation of high-voltage transmission networks in order to better support market design alternatives, price-formation mechanisms, and for general operation and planning decisions. In most cases, this process involves the definition of more complex mathematical models. Different optimization approaches based on DC Load-flow formulations are extensively used in this field River et. al (1990), Wood et al. (1996), Philpott (1999), Stott et al. (2009), Escobar and Jofre (2010).

In this paper we study the behavior of the economic dispatch formulation when it is used in operational planning and planning decision process. Our focus is on instances of these models in which negative prices are observed at some nodes. A negative price at a node indicates a situation in which the system cost can be reduced by more consumption of power at the node. When there is a free disposal of power at this node, a simple economic argument shows that the nodal price must be non-negative. So negative prices can only occur when flow balance constraints in the dispatch model are modeled as equations.

When the economic dispatch model does not include line losses, it is a linear program, which is easily solved using standard software (Ilog , 2002; Murtagh and Saunders, 1983). Linear programs are convex optimization problems, and enjoy all their desirable properties. Although negative prices may occur in these models, their existence need not be a matter of any concern. On the other hand, when losses are modeled as quadratic functions of the line flow, it is well known that (without free disposal at the nodes) the feasible region of the dispatch model is no longer convex (Philpott and Pritchard, 2004). This means that the benefits of convex optimization are no longer guaranteed.

In this paper we investigate the extent to which this loss of convexity is material in solving realistic dispatch problems. We demonstrate that non-negative prices are sufficient to guarantee global optimality, even if the feasible region is not convex, and so a negative nodal price is an indicator of a possible problem for optimization software. In particular, we show that linear programming software that approximates losses by piecewise linear functions will not represent losses properly as it becomes more accurate. Quadratic programming software that assumes positive definite Hessian matrices may also encounter problems in solving such problems. Finally nonlinear optimization software that yields at best a local optimum may not give a global optimal solution when prices are negative.

An appropriate understanding of these phenomena is useful in a planning process where thousands of economic dispatch instances must be evaluated in order to define an operational policy or to determine generation and transmission network investment. Besides, identifying and properly handling non-convex cases (indicated by a negative price) could avoid making wrong decisions on operational policies or investment.

The paper is laid out as follows. In the next section, we give some general mathematical results that can be applied to the dispatch problem. A general mathematical formulation of the economic dispatch problem is then presented in Section 3, and we present a three-busbar model to illustrate the formulation, and to serve as an example of some of the difficulties we describe. We first show in Section 4 how linear programming software might fail when applied to this problem. In Section 5 we compute the Hessian matrix of the Lagrangian with respect to the flow-balance constraints, and show how this might fail to be positive semi-definite with negative prices. In Section 6 a nonconvex six-busbar example is described and used to illustrate how a negative price might be an indicator of a local optimum solution. Finally, Section 7 presents the conclusions of the work.

2 Preliminaries

Consider the general optimization problem

$$
\begin{array}{rll}
E(u): \min & \sum_i h_i(q_i) & \\
\text{s.t.} & g_i(f) + q_i & = u_i,\ i = 1, 2, \ldots, n \\
& Af + Bq & = b, \\
& f \in F, \quad q \in Q. &
\end{array} \tag{1}
$$

where each h_i is a convex function and each g_i is a concave function; A and B are $p \times m$ and $p \times n$ real matrices; b lies in R^p; and F and Q are convex sets in R^m and R^n respectively. We define the following relaxation of $E(u)$:

$$
\begin{array}{rll}
G(u): \min & \sum_i h_i(q_i) & \\
\text{s.t.} & g_i(f) + q_i & \geq u_i,\ i = 1, 2, \ldots, n \\
& Af + Bq & = b, \\
& f \in F, \quad q \in Q. &
\end{array} \tag{2}
$$

Assume that every feasible point of $E(u)$ and $G(u)$ satisfies a constraint qualification, (see e.g. Bazaraa et al. , 2006). Let $\lambda_i(E)$, $i = 1, 2, \ldots, n$, be the Lagrange multipliers from the first set of constraints for E (and let $\lambda_i(G)$ be defined similarly).

Proposition 1 $\lambda_i(G) \geq 0$, $i = 1, 2, \ldots, n$.

Proof See Bazaraa et al. (2006).

Proposition 2 *If $\lambda_i(E) \geq 0$, $i = 1, 2, \ldots, n$ for a locally optimal solution to $E(u)$, then this is also optimal for $G(u)$.*

Proof Suppose that (q^*, f^*) solves $E(u)$, with $\lambda_i(E) \geq 0$, $i = 1, 2, \ldots, n$. Then $(q^*, f^*, \lambda(E))$ satisfies the Karush-Kuhn-Tucker conditions of $G(u)$. Since $G(u)$ is a convex programming problem these conditions are sufficient.

Corollary 1 *A locally optimal solution to $E(u)$ with $\lambda_i(E) \geq 0$, $i = 1, 2, \ldots, n$ is also globally optimal.*

Corollary 2 *Let* $\phi_E(u)$ *be the optimal value of* $E(u)$*. If* $\lambda_i(E(u)) \geq 0$, $i = 1, 2, \ldots, n$ *for every* u *then* $\phi_E(u)$ *is convex.*

Proof Follows from the convexity of the optimal value of the convex programming problem $G(u)$. See Bazaraa et al. (2006).

3 The Economic Dispatch Problem

3.1 The General Case

The economic dispatch problem for an electricity generation and transmission system is typically formulated by expressing link flows and losses in terms of voltage angles at each busbar (Wood and Wollenberg, 1996). Both power flow balances at each busbar (with or without ohmic losses representation) and transmission limits using the DC load flow approximation, represent all the transmission constraints.

$$
\begin{aligned}
Z = Min & \left\{ \sum_{i=1}^{NN} \{ \sum_{k \in \Omega_i^G} C_{Gk}(P_{Gk}) + \sum_{k \in \Omega_i^C} C_{Uk}(P_{Uk}) \} \right\} \\
& s.t. \\
& \sum_{k \in \Omega_i^G} P_{G_k} - \sum_{j \in \Omega_i^N} \left(\frac{\theta_i - \theta_j}{x_{ij}} + \frac{r_{ij} \left(\theta_i - \theta_j\right)^2}{2x_{ij}^2} \right) + \\
& \sum_{k \in \Omega_i^C} P_{U_k} = \sum_{k \in \Omega_i^C} P_{C_k}, \quad i = 1, 2, \ldots, NN \\
& \theta_i - \theta_j \leq x_{ij} \overline{F}_{ij} \; \forall (i,j) \in \Omega^L \\
& \theta_j - \theta_i \leq x_{ij} \overline{F}_{ji} \\
& \underline{\mathbf{x}_E} \leq \mathbf{x}_E \leq \overline{\mathbf{x}_E}
\end{aligned}
\tag{3}
$$

where parameters are:

- NN = number of busbars
- Ω_i^N = set of nodes adjacent to node i
- Ω_i^G = set of generators at node i
- Ω_i^C = set of demands at node i
- Ω_i^L = set of transmission lines
- C_{Gk} = convex generation cost function,
- C_{Uk} = convex cost function of unserved energy.
- x_{ij} = line series reactance expressed in per unit.
- r_{ij} = represents the equivalent resistance expressed in per unit.
- $\overline{F}_{ij}$ = maximum active power flow on transmission line between nodes i and j expressed in per unit.

and variables are:

- P_{Gk} = generation active power injection in per unit.
- P_{Uk} = unserved energy (active power) in per unit.
- θ_i = voltage phase angle from node i expressed in radians.
- P_{C_j} = active power load expressed in per unit.
- $\mathbf{x}_E = \left[P_G \; P_U \; \theta\right]$ = vector of all optimization variables.

Ohmic losses PL_{ij} for each transmission line can be obtained (River et. al , 1990) from the nonlinear function

$$PL_{ij} = 2\frac{rr_{ij}}{rr_{ij}^2 + x_{ij}^2}(1 - \cos(\theta_i - \theta_j)) \tag{4}$$

where rr_{ij} is line series resistance expressed in per unit and $r_{ij} = \frac{rr_{ij}x_{ij}^2}{rr_{ij}^2+x_{ij}^2}$.

The resulting economic dispatch problem has convex cost function, quadratic equality constraints (node balances), linear inequality constraints and bounds for each variable. It is easy to see by making the substitution $f_{ij} = \frac{\theta_i - \theta_j}{x_{ij}}$, $[P_G \; P_U] = q$, that the economic dispatch problem is in an equivalent form to $E(u)$. Here the function $g_i(f)$ takes the form of $\sum_{j \in \Omega_i^N}(-f_{ij} - \frac{r_{ij}}{2} f_{ij}^2)$. It is important to note that the economic dispatch problem can also be modeled with nonconvex generator cost curves (Chaturvedi et. al , 2008), a case that falls outside the setting we discuss here.

3.2 Example 3-busbar system

To motivate our discussion we shall study the realistic 220 kV three-busbar system shown in Fig. 1. This case study is based on realistic data and can be interpreted as a sub-network of more extended power system. Here, for each line we consider reactances $x_{ij} = 0.4 \; \Omega/km$ and equivalent resistances $r_{ij} = 0.04 \; \Omega/km \rightarrow rr_{ij} = 0.040408 \; \Omega/km$. Using a reference power Sb = 100 MVA, the resulting reactances and resistances in per unit are shown in Fig. 1.

- Line lengths: Line 1-2 = Line 1-3 = 121 km, Line 2-3 = 181.5 km.
- The transmission capacity of Line 2-3 expressed in active power is set to $\overline{F}_{23} = 50 \; MW$.
- Generation costs: Reservoir = Gen 1 $\rightarrow C_1 = 1$ \$$/MWh$ (strategic value of stored water), Gen 3 $\rightarrow C_3 = 50$ \$$/MWh$.
- Load 2 = 10 MW and Load 3 = 200 MW.
- Unserved energy costs of 500 \$$/MWh$ for each load.

The optimal economic dispatch for this example can be computed using nonlinear programming software such as MINOS (Murtagh and Saunders, 1983) and summarized as follows:

- $P_{G1}^* = 196.84 \; MW$, $P_{G3}^* = 15.73 \; MW$.
- Total Losses = 2.57 $MW \rightarrow 1.22\%$.
- System costs = 983.5 \$$/h$.

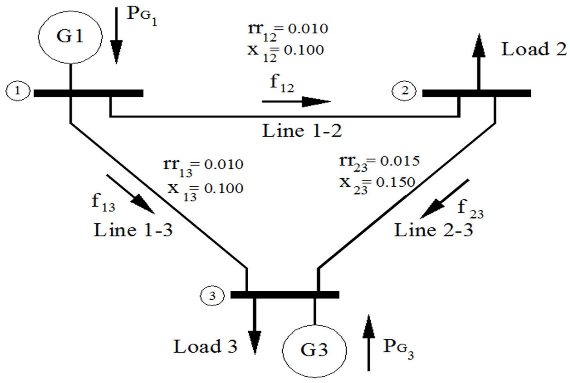

Fig. 1 Three busbar system

- $f_{12} = 60.37\ MW$, $f_{13} = 135.37\ MW$, $f_{23} = 50\ MW$.
- $\lambda_1 = 1\ \$/MWh$, $\lambda_2 = -47.591\ \$/MWh$, $\lambda_3 = 50\ \$/MWh$.

The Lagrange multipliers λ_i relate to the busbar-balance constraints and give the nodal prices at optimality, one of which λ_2 is negative. The software can be shown to terminate at a locally optimal solution, but since $\lambda_2 < 0$, we have no guarantee that this is a globally optimal solution, although this can be verified in this example by exhaustively checking the Karush-Kuhn-Tucker conditions. Recall that optimization problems of the form $E(u)$ are not convex programming problems. In the next section it is shown that further analysis is needed to define global optimality.

4 Piecewise Linear Approximation of Losses

As shown in Proposition 2, if a locally optimal solution to the economic dispatch problem has non-negative Lagrange multipliers at optimality, then this solution is also a (global) solution to the (convex) inequality constrained problem, and so it is a global solution to the dispatch problem. The fact that the solution is also a solution to $G(u)$ means that we may approximate $G(u)$ by a linear program (say $L(u)$) and expect that the solution to $L(u)$ is close to the global optimum of $G(u)$ and hence $E(u)$. Many economic dispatch systems (see e.g. Alvey et al. , 1998) use linear programming in this way.

The linear programming approximation of losses can fail when the optimal solution to $E(u)$ has a negative Lagrange multiplier as in our example (this fact is well known in the optimal power flow modelling community, see e.g. De la Torre and Galiana (2005). To illustrate this we solved the three-busbar example using following step piece-wise linear loss functions:

$$P_{Lij} = \sum_{k=A}^{C} m_{ij}^{k}\left(\Delta\theta_{ij}^{k}\right); \qquad \theta_i - \theta_j = \sum_{k=A}^{C} \Delta\theta_{ij}^{k}. \tag{5}$$

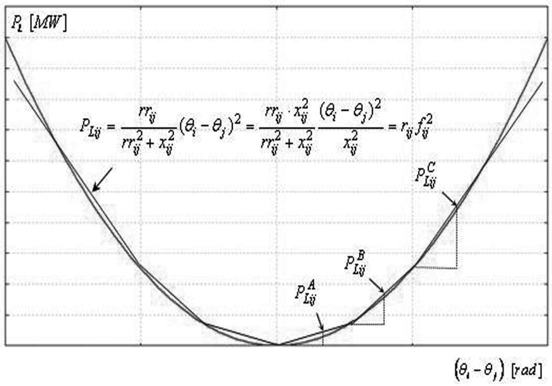

Fig. 2 Ohmic losses representation

Considering three step piece-wise linear, that is, $k = A, B, C$, the loss model in the three-busbar system can be writen as

$$
\begin{aligned}
P_{L12} &= 100\,(\theta_1 - \theta_2)^2 \approx P_{L12}^A + P_{L12}^B + P_{L12}^C \\
P_{L13} &= 100\,(\theta_1 - \theta_3)^2 \approx P_{L13}^A + P_{L13}^B + P_{L13}^C \\
P_{L23} &= 66.\overline{6}\,(\theta_2 - \theta_3)^2 \approx P_{L23}^A + P_{L23}^B + P_{L23}^C
\end{aligned}
$$

Fig. 2 shows the ohmic losses representation by three linear functions.

Solving the resulting optimization problem for three loss function levels in each transmission line we obtained the following loss values in MW:

$$
\begin{array}{lll}
P_{L12}^A = 0.0000 & P_{L12}^B = 0.2709 & P_{L12}^C = 0.9863 \\
P_{L13}^A = 0.1306 & P_{L13}^B = 0.745 & P_{L13}^C = 0.9679 \\
P_{L23}^A = 0.0862 & P_{L23}^B = 0.3065 & P_{L23}^C = 0.0000
\end{array}
$$

Fig. 3 shows the optimal result achieved for the ohmic losses in Line 1-2 (between nodes 1 and 2).

It can be observed that the optimization arrives at an infeasible solution from the physical point of view. The software tries to maximize losses in Line 1-2 using the second and third level of the loss function P_{L12}^B, P_{L12}^C. The correct physical losses are 0.3698 MW, while the ohmic losses based on the linear approximation are 1.2572 MW, i.e. 3.4 time bigger. In this way, more power can be allocated by the cheaper generation at busbar one. So the piecewise linear approximation has failed to represent the problem in the way we intended.

We might have expected some problems here since we are approximating a nonconvex optimization problem with a convex one. Indeed the example has a negative price so we do not have a guarantee that $E(u)$ has the same solution as $G(u)$ its convexification. We proceed to show under fairly mild conditions on the optimal solution that a negative price at any node indicates that a linear programming model of the form above will give an incorrect flow

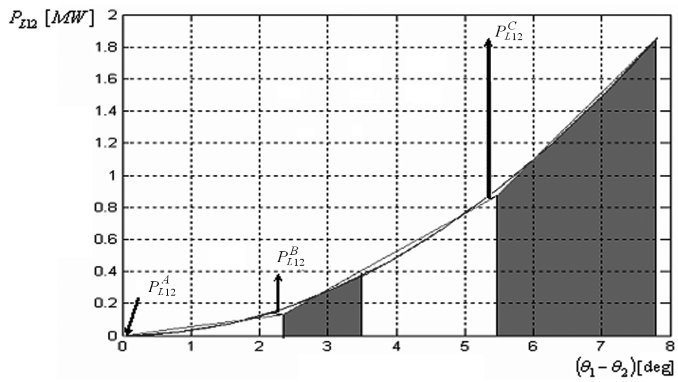

Fig. 3 Ohmic losses in line 1-2 with piecewise linear loss functions. The shaded areas show where P_{L12}^{B} >0 and P_{L12}^{C} >0.

representation if there are enough pieces in the piecewise linear representation of the loss curves.

Our result is stated in the framework of the problem $E(u)$, for a transmission network of directed lines ij, $i < j$, where

$$g_i(f) = \sum_{j<i}(f_{ji} - \frac{r_{ji}}{2}f_{ji}^2) + \sum_{j>i}(-f_{ij} - \frac{r_{ij}}{2}f_{ij}^2),$$

and for each transmission line ij, $-F_{ij} < f_{ij} < F_{ij}$. We assume that $r_{ij} > 0$ for all lines ij implying that $\frac{r_{ij}}{2}f_{ij}^2$ is a strictly convex function. In the linear program, we model $l_{ij} = \frac{r_{ij}}{2}f_{ij}^2$ as piecewise linear convex functions, where

$$\begin{array}{ll} f_{ij}^{\nu} = -F_{ij} + y_{ij}^1 + y_{ij}^2 + \ldots + y_{ij}^{\nu}, & i < j, \\ 0 \le y_{ij}^k \le b_{ij}^k,\ k = 1, 2, \ldots, \nu, & i < j, \\ l_{ij}^{\nu} = \frac{r_{ij}}{2}F_{ij}^2 + m_{ij}^1 y_{ij}^1 + m_{ij}^2 y_{ij}^2 + \ldots + m_{ij}^{\nu} y_{ij}^{\nu}, & i < j, \end{array}$$

where the strict convexity assumption gives increasing slopes, i.e.

$$m_{ij}^k < m_{ij}^{k+1},$$

and each slope is in the interval $(-1, 1)$, since the marginal loss can never exceed the marginal flow.

Our result requires the following definition.

Definition 1 A dispatch is *degenerate* if there exists a node i, at which every generator is either not dispatched or fully dispatched, and all lines ij have flow at an upper bound or a lower bound. (The dispatch computed in the previous 3-node example is not degenerate.)

Proposition 3 *Suppose the optimal dispatch is not degenerate. Any node i with a negative price in the optimal dispatch has $q_k = 0$, $k \in \Omega_i^G$ and at least one incident line with flow between its bounds.*

Proof Consider node i with a negative price $\lambda_i < 0$ for some optimal dispatch. Since the dispatch is optimal, and generator costs are non-negative, the Karush-Kuhn-Tucker conditions imply that every generator at node i is dispatched zero. Since the dispatch is not degenerate, then there is some line ij with $-F_{ij} < f_{ij} < F_{ij}$, or some line ji with $-F_{ji} < f_{ji} < F_{ji}$, i.e. i has at least one incident line with flow between its bounds.

In our example, the line 1-2 connecting buses 1 and 2 has power flowing into node 2 at a rate less than the line's capacity, as predicted by the proposition. Proposition 3 implies that any node i with a negative price and positive demand in a nondegenerate optimal solution has some positive flow entering i from some other node j. On the other hand, a must-run generation plant with fixed output (such as a nuclear plant) could amount to a fixed negative demand at i, with positive flow leaving i.

The following proposition considers a sequence LP^ν of linear programming problems each of which approximates $E(u)$ using piecewise linear approximations of the quadratic loss functions for each line as described above. Thus in LP^ν each line flow is modelled by a sum of ν nonnegative flow variables $y_{ij}^k \leq b_{ij}^k$ with strictly increasing constant marginal losses. We denote an optimal solution of LP^ν by (q^ν, f^ν) and let λ^ν be the corresponding nodal prices. We say that the flow f^ν *correctly represents the losses* if for each line flow $f_{ij}^\nu = -F_{ij} + y_{ij}^1 + y_{ij}^2 + \ldots + y_{ij}^\nu$ and each $k > 1$ we have $y_{ij}^k > 0$ implies $y_{ij}^{k-1} = b_{ij}^{k-1}$.

Proposition 4 *Suppose demand at each node is nonnegative, the optimal dispatch (q, f) for $E(u)$ is nondegenerate with prices λ, and some node has a negative price. Suppose there is some N such that for all $\nu > N$, the optimal flow f^ν that solves LP^ν correctly represents the losses. Then $(q^\nu, f^\nu, \lambda^\nu)$ does not converge to (q, f, λ).*

Proof Let i be the node with a negative price in the optimal solution. By proposition 3 there is no generation at i, and at least one line ji with $-F_{ji} < f_{ji} < F_{ji}$. (If necessary, we can rename the node indices to make $j < i$.) Suppose $(q^\nu, f^\nu, \lambda^\nu) \to (q, f, \lambda)$. Then for sufficiently large ν we have $\lambda_i^\nu < 0$ and

$$-F_{ji} < f_{ji}^\nu < F_{ji}.$$

The flow

$$f_{ji}^\nu = -F_{ji} + y_{ji}^1 + y_{ji}^2 + \ldots + y_{ji}^\nu$$

is the sum of variables y_{ji}^k from ν loss sections, and by assumption $y_{ji}^k = b_{ji}^k$ for all sections up to the one corresponding to f_{ji}^ν, and $y_{ji}^k = 0$ for the sections beyond. (We can make ν large enough so the section k corresponding to f_{ji}^ν is between 2 and $\nu - 1$.)

Let ρ^k be the reduced cost of each of these variables at optimality of the problem LP^ν. Then it is easy to show that

$$\rho^k = (\lambda_i^\nu + \lambda_j^\nu) m_{ji}^k + (\lambda_j^\nu - \lambda_i^\nu)$$

where m^k is the slope for loss section k. We have

$$(\lambda_j^\nu - \lambda_i^\nu) > (\lambda_j^\nu + \lambda_i^\nu)$$

so if $(\lambda_j^\nu + \lambda_i^\nu) \geq 0$, then

$$\rho^k > (\lambda_i^\nu + \lambda_j^\nu)(1 + m_{ji}^k) \geq 0$$

for all k, and so $f_{ji}^\nu = -F_{ji}$, yielding a contradiction. Thus we have

$$\lambda_j^\nu + \lambda_i^\nu < 0.$$

Now $m_{ji}^k < m_{ji}^{k+1}$ implies that

$$\begin{aligned}\rho^k &= (\lambda_i^\nu + \lambda_j^\nu)m_{ji}^k + (\lambda_j^\nu - \lambda_i^\nu) \\ &> (\lambda_i^\nu + \lambda_j^\nu)m_{ji}^{k+1} + (\lambda_j^\nu - \lambda_i^\nu) \\ &= \rho^{k+1}\end{aligned}$$

Since $-F_{ji} < f_{ji}^\nu < F_{ji}$, the assumption that (q^ν, f^ν) solves LP^ν gives $\rho^1 \leq 0$ and $\rho^\nu \geq 0$, contradicting $\rho^k > \rho^{k+1}$.

5 Quadratic Optimization of Line Losses

The failure of linear programming in these circumstances points to the use of optimization software that will compute optimal solutions with quadratic functions. The quadratic terms from the losses can be placed in a Lagrangian, where the Lagrange multipliers λ_i are chosen to be those that pertain at the global optimal solution (We shall assume regularity conditions that ensure these exist). The Lagrangian for the economic dispatch problem can then be expressed as

$$\begin{aligned}\mathcal{L}(P_G, P_U, \theta) = &\sum_{i=1}^{NG} C_{Gi}(P_{Gi}) + \sum_{i=1}^{ND} C_{Ui}(P_{Ui}) \\ &+ \sum_{i=1}^{NN} \lambda_i [\sum_{j \in \Omega_i^C} P_{C_j} - \sum_{j \in \Omega_i^G} P_{G_j} + ... \\ &\sum_{j \in \Omega_i^N} \left(\frac{\theta_i - \theta_j}{x_{ij}} + \frac{r_{ij}\,(\theta_i - \theta_j)^2}{2x_{ij}^2} \right) - \\ &\sum_{j \in \Omega_i^C} P_{Ui}] \end{aligned} \tag{6}$$

It can be observed that for positive multipliers λ, the Lagrangian is convex in P_G, P_U, and θ. For linear/quadratic functions $C_{Gi}(\cdot)$ and $C_{Ui}(\cdot)$, $\mathcal{L}(P_G, P_U, \theta)$ is easily seen then to be a positive semi-definite quadratic form.

If some Lagrange multiplier λ_i is negative, then $\mathcal{L}(P_G, P_U, \theta)$ may not be positive semi-definite. To investigate this we will compute its explicit form. Let A be the adjacency matrix of the network

$$a_{ij} = \begin{cases} 1 & \text{busbar } i \text{ and } j \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
\mathcal{L}(P_G, P_U, \theta) &= \sum_{i=1}^{NN} \lambda_i \sum_{j=1}^{NN} a_{ij} \left(\frac{\theta_i - \theta_j}{x_{ij}} + \frac{r_{ij} \left(\theta_i - \theta_j\right)^2}{2x_{ij}^2} \right) .. \\
&= \sum_{i=1}^{NN} \sum_{j=1}^{NN} \lambda_i a_{ij} \left(\frac{\theta_i - \theta_j}{x_{ij}} + \frac{r_{ij} \left(\theta_i - \theta_j\right)^2}{2x_{ij}^2} \right) .. \\
&= \sum_{i=1}^{NN} \sum_{j=1}^{NN} \lambda_i a_{ij} \frac{\theta_i - \theta_j}{x_{ij}} + \\
&\quad + \sum_{i=1}^{NN} \sum_{j=1}^{NN} \lambda_i a_{ij} \frac{r_{ij} \left(\theta_i - \theta_j\right)^2}{2x_{ij}^2} + ..
\end{aligned} \tag{7}$$

The Hessian H of the Lagrangian is defined by

$$\begin{aligned}
H_{ii} &= \sum_{j=1}^{NN} \lambda_i a_{ij} \frac{r_{ij}}{x_{ij}^2} + \sum_{j=1}^{NN} \lambda_j a_{ji} \frac{r_{ji}}{x_{ji}^2} \\
H_{ij} &= -\lambda_i a_{ij} \frac{r_{ij}}{x_{ij}^2} - \lambda_j a_{ji} \frac{r_{ji}}{x_{ji}^2}
\end{aligned} \tag{8}$$

Let $\sigma_{ij} = \sigma_{ji} = a_{ij} \frac{r_{ij}}{x_{ij}^2}$. Then

$$\begin{aligned}
H_{ii} &= \sum_{j=1}^{NN} (\lambda_i + \lambda_j) \sigma_{ij} \\
H_{ij} &= -(\lambda_i + \lambda_j) \sigma_{ij}
\end{aligned} \tag{9}$$

Observe that

$$\sum_j H_{ij} = \sum_i H_{ij} = 0$$

and so H is singular. Also observe that

$$\begin{aligned}
\sum_{i \neq k} \sum_{j \neq k} H_{ij} &= \sum_{i \neq k} \sum_{l=1}^{NN} (\lambda_i + \lambda_l) \sigma_{il} + \\
&\quad \sum_{i \neq k} \sum_{j \neq k} -(\lambda_i + \lambda_j) \sigma_{ij} \\
&= \sum_{i \neq k} (\lambda_i + \lambda_k) \sigma_{ik} = H_{kk}
\end{aligned} \tag{10}$$

In the three-busbar example

$$(\sigma_{ij}) = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & \frac{2}{3} \\ 1 & \frac{2}{3} & 0 \end{bmatrix}$$

so

$$H = 100 \begin{bmatrix} 2\lambda_1+\lambda_2+\lambda_3 & -\lambda_1-\lambda_2 & -\lambda_1-\lambda_3 \\ -\lambda_1-\lambda_2 & \lambda_1+\frac{5}{3}\lambda_2+\frac{2}{3}\lambda_3 & -\frac{2}{3}(\lambda_2+\lambda_3) \\ -\lambda_1-\lambda_3 & -\frac{2}{3}(\lambda_2+\lambda_3) & \lambda_1+\frac{2}{3}\lambda_2+\frac{5}{3}\lambda_3 \end{bmatrix}$$

Recall that $\lambda_1 = 1$, $\lambda_2 = -47.591$, $\lambda_3 = 50$, so $\lambda_1 + \frac{5}{3}\lambda_2 + \frac{2}{3}\lambda_3 < 0$, which means that H is indefinite for these choices of λ. This will cause problems for quadratic programming solvers (e.g. some interior point methods) that require at least positive semi-definite Hessian matrices. It is interesting to observe, however, for this example that H restricted to the tangent plane of the active line capacity constraint

$$100\left(\frac{\theta_2 - \theta_3}{0.15}\right) = 50$$

gives a reduced Hessian

$$H_r = 100 \begin{bmatrix} 2\lambda_1+\lambda_2+\lambda_3 & -(2\lambda_1+\lambda_2+\lambda_3) \\ -(2\lambda_1+\lambda_2+\lambda_3) & 2\lambda_1+\lambda_2+\lambda_3 \end{bmatrix} \quad (11)$$

that is positive semi-definite as long as $2\lambda_1 + \lambda_2 + \lambda_3 \geq 0$. Thus a reduced gradient algorithm that identified this active constraint would not have to deal with an indefinite Lagrangian. The solution computed in section 3 was found using the reduced gradient nonlinear optimization package MINOS (Murtagh and Saunders, 1983).

6 Nonconvex Six-Busbar Example

In Section 4 it was demonstrated that a negative price leads to the failure of linear programming approximations of the dispatch model, that to some extent is overcome by nonlinear programming algorithms. Of course the problem $E(u)$ is not convex, and so we have no guarantee that the nonlinear programming system will locate the global optimum. To illustrate this, consider two identical power exchanges linked by a transmission line with ohmic losses in nodes with negative marginal prices. For this analysis we used twice the previous three busbar example interconnecting both systems at their respective busbar 2 (see Fig. 4).

The optimal economic dispatch after they are connected is not symmetric. In order to allocate more power from the cheap generation at generator $G1$, energy is transferred from the upper to the lower system (15.85 MW). This is an increase of load at Busbar 2 that reduces the dispatch of the expensive generator $G3$ to zero. In the lower system we observe the opposite behavior. Nevertheless, the final result is cheaper than twice the costs of the operation

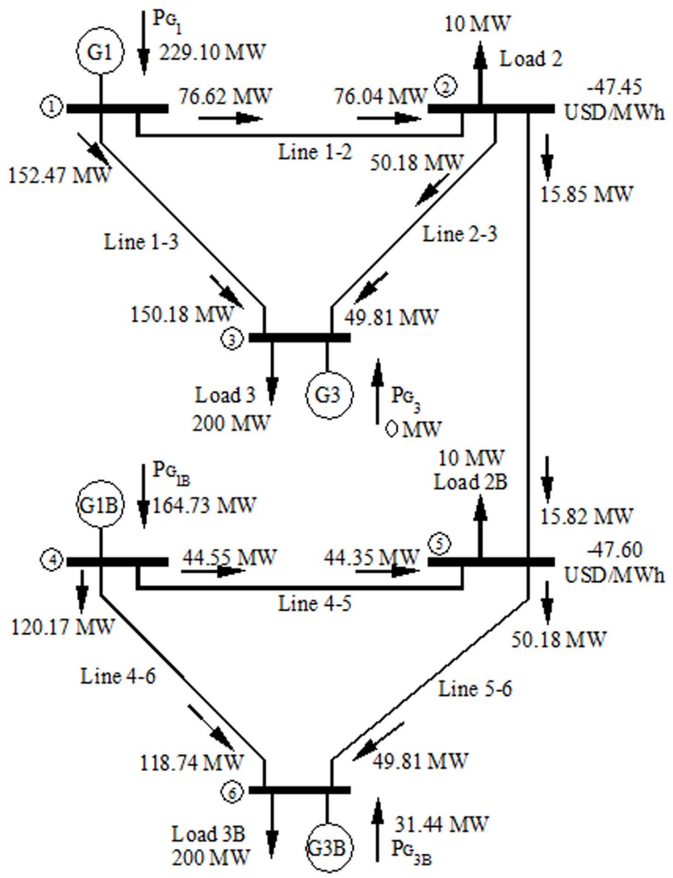

Fig. 4 Two symmetrical systems/markets

of two disconnected networks ($2 * 983.5 = 1967.0 > 1965.99$ $\$/h$). When given a starting point equal to the optimal dispatch in each separate system, and no flow in the connecting line, MINOS terminates at this local optimal solution.

7 Conclusions

In this paper we have discussed the behavior of economic dispatch models with ohmic losses when they are used in transmission constrained economic dispatch and nodal pricing models. We have shown that negative nodal prices at the optimal dispatch solution could indicate convergence problems for a convex optimization algorithm. Indeed, a negative nodal price at a non degenerate solution of the transmission constrained economic dispatch problem could imply losing convexity and then any linear or piecewise linear approximation will fail to converge as the loss representation becomes more accurate. Moreover in complicated transmission networks, a negative price could indicate that only a local optimal solution has been found.

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References

River M, Pérez-Arriaga IJ, Luengo G (1990) A model for computation of spot prices in interconnected power systems. 10th PSCC Conf., Graz,Austria

Wood A, Wollenberg B (1996) Power Generation, Operation and Control. Wiley, New York

Philpott A (1999) Experiments with load flow pricing models. CRNEC Policy Conference, Auckland, New Zealand

Stott B, Jardin J and Alsac O (2009) DC power flow revisited. IEEE Trans. Power Syst., 24: 1290-1300

Escobar J and Jofre A (2010) Monopolistic competition in electricity networks with resistance losses. Economic Theory, 44: 101-121

Ilog (2002) CPLEX 8.1 Reference Manual

Murtagh B and Saunders M (1983) MINOS 5.0 User's guide. Systems Optimization Laboratory Dept. of Operations Research, Stanford University, Tech. Rep. (83-20)

Philpott A and Pritchard G (2004) Financial transmission rights in convex pool markets. Operations Research Letters, 32: 109-113

Bazaraa M, Sherali H, Shetty C (2006) Nonlinear Programming, Theory and Algorithms. Wiley, New York

Chaturvedi K, Pandit M and Srivastava L (2008) Self-organizing hierarchical particle swarm, optimization for nonconvex economic dispatch. IEEE Trans. on Power Syst., 23: 1079-1087

Alvey T, Ma X and Sun D (1998) A security-constrained bid-clearing system for the New Zealand wholesale electricity market. IEEE Trans. on Power Syst., 13: 340-346

De la Torre S and Galiana F (2005) On the Convexity of the System Loss Function. IEEE Trans. on Power Syst., 20: 2061-2069