Submitted to *Operations Research*
manuscript (Please, provide the mansucript number!)

Electricity Contracting and Policy Choices under Risk-aversion

Anthony Downward
Department of Engineering Science, University of Auckland, New Zealand, a.downward@auckland.ac.nz

David Young
Electric Power Research Institute, Palo Alto. This work completed whilst a Postdoctoral Fellow at the University of Auckland Energy Centre, dyoung@epri.com

Golbon Zakeri
Department of Engineering Science, University of Auckland, New Zealand, g.zakeri@auckland.ac.nz, http://www.epoc.org.nz

Much of the literature on electricity markets has been predicated on the assumption that firms are risk neutral; that they make decisions based upon maximizing expected profits. In reality, many firms are controlled by shareholders or owners who take a point of view usually no longer than five to ten years hence, and seek to minimize risk. A firm that does not return profits in that time period is likely to be wound up or forced into bankruptcy. Under such an incentive scheme, firms are likely to be risk-averse, with serious implications for policy analysis. We introduce a model of an electricity market where firms can choose to enter the retail market, then enter into retail contracts, and finally purchase electricity in a wholesale market to satisfy their contracts. We explicitly assume that firms are risk-averse in this model, and act to minimize their conditional value-at-risk (while maximizing profit). We demonstrate how firms' behaviour changes with risk-aversion, and use the example of an asset-swap policy to demonstrate the importance of risk-aversion in determining policy outcomes.

Key words: risk-aversion, game thoery, retail pricing, electricity

1. Introduction

Much of the literature on electricity markets is based on published models that assume firms act in a risk-neutral manner. Classic results such as the importance of long-term contracts in lowering spot market prices (Allaz and Vila, 1993; Carlton, 1979) and the efficiency of nodal pricing markets (Caramanis, 1982) were proved using this assumption. However, the question of whether risk-neutrality is a realistic assumption is open to attack. A risk-neutral firm, for instance, would be happy to take on a 99% chance of making zero profits if the last 1% chance earned a profit sufficiently large. Such a firm would theoretically be perfectly happy operating over a hundred year period making no money year to year. In practice, the firm's investors would likely pull the plug long before the hundred years were up. If firms are in reality risk-averse then many of the policy predictions are suspect. For example, Neuhoff and De Vries (2004) show that generation investment is not necessarily efficient in equilibrium if consumers and investors are risk-averse.

Much of the risk inherent in electricity markets falls on the retail sector. Retailers typically form fixed price, variable quantity contracts with their customers (residential and commercial consumers). This places all of the risk of supplying the electricity on the retailer, exposing them to a variety of sources of uncertainty through the wholesale market, including price fluctuations from transmission or plant outages, fuel cost shocks, or direct risk from demand shocks. In many markets, retailers vertically integrate with generators. This is presumably to mitigate the risk faced by the retailer, yet there is almost no analysis on this subject in the literature.[1] The few papers that explicitly use risk-averse firms usually focus on long-term contract formation by generators and avoid modelling the retail market. There is also no discussion of transmission risk, which is arguably critical to pricing in nodal markets, and limited discussion of any kind of competition or entry into retail markets. Furthermore, all of these papers employ the expected utility formulation of risk-aversion, which abstracts away from the behaviour of real-world firms.

In this paper, we introduce a model of an electricity market where risk-averse generators and retailers act to maximize profit in a three-stage game. We explicitly model the formation of contract prices in the retail sector, allowing for full strategic behaviour by firms. Furthermore, firms have the option of choosing whether or not to enter the retail market in the first place (including those firms who own generation), making vertical integration (or lack thereof) an endogenous outcome of the model. The other novel feature in our model is our characterization of risk-aversion. Rather than pick an arbitrary expected utility function, the firms in our model maximize profit while minimizing conditional value-at-risk (henceforth CVaR). This measure is similar to value-at-risk

[1] This can also hedge generators where fuel costs are uncertain.

(VaR) and thus arguably closer to capturing how real world firms actually behave in response to uncertainty.

The existing literature on risk-aversion in electricity markets can be crudely divided into three distinct strands. One is the modelling of the decisions of individual generators based upon input risks (such as water flows for hydro generators). This considers risk from the perspective of a single generator, and does not consider market outcomes of multiple risk-averse generators, so is not particularly relevant to our model. Another strand takes the point of view of large consumers faced with a risky series of electricity prices. In this strand, prices are taken as exogenous, not formed through competition between risk-averse firms. The final strand, most relevant to this paper, is the study of risk-aversion in the formation of long-term contracts. Examples include articles by Powell (1993), Neuhoff and De Vries (2004), and Baldursson and von der Fehr (2007). These papers all use the expected utility method to model risk-aversion, since concave utility functions are known to mimic risk-averse behaviour. Of these, only Baldursson and von der Fehr analyze the behaviour of risk-averse firms in both wholesale and retail markets simultaneously. They consider a model where risk-averse firms vertically integrate, finding that vertical integration impairs market performance by increasing the gap between contract and (expected) spot prices. Their measure of risk is mostly the constant relative risk-aversion (CRRA) measure commonly used by economists.[2] Firms are price takers in the wholesale (spot) market. They consider either fixed retail prices, or prices linked to wholesale prices. Retail prices are not determined by retail competition. The goal of their paper is to see if long-term contracts can still enhance competition when risk-aversion is incorporated.

A limitation of the existing literature is the use of expected utility functions to model risk-aversion – i.e. assuming that the expected utility from a given strategy is less than the utility of the expected profit of that strategy. In practice, firms often use the concept of value-at-risk to measure their exposure to risk. Value-at-risk is ubiquitous throughout finance: both in the academic and practical sides of the field. Here risk is defined as the threshold value that the losses may exceed

[2] They use a more general formulation in their initial modelling.

with some probability. For example if losses are predicted to exceed \$100 with a probability of 5% then the $\text{VaR}_{5\%}$ is \$100. Investors with a high level of risk-aversion have the option of minimising their value-at-risk, trading off low returns for little or no chance of losing money. Investors with less risk-aversion can do the opposite. An economic model directly incorporating the assumption of value-at-risk can be reasonably argued to be more realistic than those that assume either risk-neutrality or incorporate risk through the use of a utility function.

Value-at-risk has not been used much in equilibrium models due to its intractability. In particular, utility functions incorporating value-at-risk need not be convex (Artzner et al., 1999). However, *conditional value-at-risk* (CVaR) is closely related, and has been employed to model generator behaviour in the Operations Research literature. Papers in this area include Carrion et al. (2009) and Carrion (2008). All these papers model a single firm's point of view; considering the best response of the firm given perceived risk. This paper, on the other hand, considers the equilibrium responses of multiple firms acting to maximize their profits less conditional value-at-risk. We demonstrate that the behaviour of risk-neutral firms compared to firms maximizing their profit less conditional value-at-risk can be very different, with obvious consequences for policy decisions.

In this paper we begin by presenting the concept of conditional value-at-risk and discussing how we employ it in the context of electricity markets. We then construct a model of an electricity market with three stages: entry, retail competition to form retail contracts, followed by wholesale market-clearing to set the spot prices.[3] We use this model first to present a simple example to illustrate the impact of risk-aversion on retail pricing and finally a general model formulation which we use to show how nodal price risk can cause generators to limit their retail obligations to nodes where they have their own generation capacity. We give examples that illustrate how risk-averse firms' behaviour varies significantly from their risk-neutral counterparts, and finish with an extended policy example: an analysis of a generator asset-swap in the New Zealand Electricity Market, designed to reduce risk and thereby decrease retail prices. Our model has several features

[3] We assume a competitive equilibrium in the wholesale market.

that set it apart from the literature to date. Not only do we use an innovative measure of risk, but we explicitly incorporate the option for firms (both those owning generators and new entrants) to enter the retail market. This implies the model could be used to discuss concepts of vertical integration, as well as the formation of long-term contracts, and can be used to analyze market design issues around such concepts.

2. Modelling Risk-averse Firms

When companies are making investment decisions or entering contracts they do not solely consider the expected benefit from the decisions. They also take into account the consequences if the return on the investment is lower than expected – behaviour which is known as risk-aversion. Companies are responsible to shareholders over both the short and long term, so they need to make decisions that seek to minimize their risk, while at the same time maximizing expected profit.

In the academic literature, there are several commonly used methods to model risk-aversion. One common way of incorporating risk-aversion into optimization problems is to use a utility function. This approach is commonly used by economists. If a business is risk-averse, a concave, increasing utility function will reflect this. That business can then maximize its expected utility (as a function of profit) rather than its expected profit; the optimal solution to this will have a lower (or equal) expected return than if the firm had been profit maximizing, but the risk associated with this decision will be less. In the finance literature, mean-risk optimization is commonly used, in portfolio optimization. This approach involves solving a bi-objective optimization problem and typically results in finding a set of Pareto optimal solutions, known as the *efficient frontier*. In such models risk is typically measured by the variance of the return or by using the downside-risk, as introduced by Markowitz (1959). A popular measure of risk is *value at risk* (VaR). Here risk is defined as the threshold value that the losses may exceed with some probability. For example if losses are predicted to exceed \$100 with a probability of 5% then the $\mathrm{VaR}_{5\%}$ is \$100.

In order to formalize the concept of risk, Artzner et al. (1999) introduced *coherent* measures of risk. A coherent risk measure must comply with the following four axioms: sub-additivity;

translation invariance; positive homogeneity; and monotonicity. Risk measures complying with these axioms exhibit key properties that are valuable for a risk-averse agent. For example, sub-additivity ensures that there is a risk-pooling effect: the sum of risks is greater than or equal to the risk of the sum. Critically, none of the above risk measures satisfy the property of coherency. The Value at Risk measure, for example, violates the sub-additivity property.

The current approaches are unsatisfactory in other ways. The utility function approach is the one used by previous papers in this literature. However, determining a particular utility function is difficult. There are several common utility functions used to model risk in the economics literature, such as the constant risk-aversion utility function and the natural log function, but these are chosen primarily for tractability, not for realism. Of the above, Value at Risk would be the most attractive choice from a realism point of view, as it is widely used by firms as well as by academics. However, it is very intractable to use in an optimization setting, since its violation of the sub-additivity property can lead to a lack of convexity.

However, there is a related measure known as *conditional value at risk* (CVaR)[4] that is a coherent risk measure. The CVaR at level α of some random return z is simply the expected loss if one's interest were restricted to the lowest $100\alpha\%$ of returns. If returns z are continuously distributed with some distribution function $F(z)$ and associated probability density function $f(z)$ then $\mathrm{CVaR}_\alpha(z)$ can be written as:

$$\mathrm{CVaR}_\alpha\left(z\right) = -\frac{1}{\alpha}\int_{-\infty}^{F^{-1}(\alpha)} z f\left(z\right) dz.$$

Initially this measure of risk may seem impractical for optimization problems. The above description requires the order of the profits to be known a priori, when in fact, in many circumstances, the ordering of outcomes often will depend on the decision variables being optimized over. Fortunately though, Shapiro et al. (2009) present an alternate formulation which enables the bottom $100\alpha\%$ of outcomes to be computed through a linear program. Moreover, for profit functions that are concave in the decision variables, we have a convex optimization problem when we maximize a weighted

[4] This is also known as average value at risk, or expected shortfall.

combination of expected profit and risk. In the formulation below, θ is a parameter between 0 and 1 and changes the weightings on risk and return[5], z is a random variable which may be a function of some of the parameters.

$$\max \qquad (1-\theta)\, E_z\,[z] - \theta \mathrm{CVaR}_\alpha\,(z)\,. \tag{1}$$

In this paper, we use the above mean-risk formulation in our model of an electricity retail market with risk-averse firms.

3. Electricity Market Model with One Node

In this section we begin our exposition by outlining the model in the single node case. Assume that electricity is traded in a market with two stages. At the first stage, retailers offer electricity contracts for sale to consumers. These are fixed price, variable quantity contracts, by which the retailer agrees to supply any quantity of electricity to the consumer at a fixed price. Each retailer sets a price for their contracts, and consumers choose which retailer will supply them. At the second stage, retailers must buy electricity in the wholesale market to satisfy their contractual obligations. Electricity is supplied by generators who make offers to sell electricity in the wholesale market. When the market clears, the electricity is generated and delivered to consumer. Retailers pay the wholesale price for the electricity consumed by their customers, generators receive the wholesale price for the quantity of electricity dispatched. Final consumers cannot buy electricity directly on the wholesale market.

Assume that there is a set of firms $\mathcal{F}_g$ who own generation. Each firm $f \in \mathcal{F}_g$ owns a mix of two types of plant, hydro plants and thermal plants. The two types of plant differ by their short-run marginal cost functions.[6] We also allow for another set of firms who do not own any generation, but may be interested in buying electricity in the wholesale market and on-selling it in the retail market. We call these firms 'retailers' and denote the set of such firms by $\mathcal{F}_r$.

[5] See the Appendix, or Shapiro et al. (2009) for a derivation of this expression.

[6] We will specify exact cost functions later when we create our examples.

Before the actual electricity market runs, there is a 'zeroth' stage at which each firm $f \in \mathcal{F}_g \cup \mathcal{F}_r$ must decide whether or not to *enter the retail market*. We will assume that a firm will enter the retail market if its risk-adjusted profits minus the cost of entry are positive. Denote the set of generators who enter the retail market by $\mathcal{E}_g \subseteq \mathcal{F}_g$. Denote the set of pure retailers who enter the retail market by $\mathcal{E}_r \subseteq \mathcal{F}_r$. Let $\mathcal{E} = \mathcal{E}_g \cup \mathcal{E}_r$ be the set of all firms who have entered the retail market.

Before we formally define the full game, we detail behaviour in the retail and wholesale markets.

3.1. Retail Market

The first stage of the electricity auction process is the formation of retail contracts, in what we term the *Retail Market*. We assume that retail contracts are formed according to take-it-or-leave-it fixed price offers. Every firm $f \in \mathcal{E}$ simultaneously offers a price $p_f \geq 0$ at which they are willing to sell contracts. The contracts on offer are fixed price, variable quantity contracts – the retailer agrees to supply any amount of demand in real-time (conclusion of stage 2 of our model) at a fixed price determined now in stage 1.

On the demand side, we assume there is a continuum of consumers who each demand the same amount of electricity, adding up to a total demand of X MW. Each consumer must pick a firm to buy their electricity retail contract from. If consumers could see all the prices, and had no other basis on which to choose a firm, then the Bertrand assumption would hold and all consumers would choose the firm with the lowest price. We do not make this assumption as we find it less than realistic.[7] Rather, we assume that consumers have some type of preferences over which firm supplies them. These preferences might be formed from a consumer search model, or a spatial model, but in this paper we abstract away from the exact specification and note that, following Vives (1999), the general concept can be captured by assuming a differentiated products demand function.

[7] In the New Zealand market, firms offer retail contracts at fixed prices. These prices are available at the well-known PowerSwitch website. Note that there is considerable price dispersion amongst the offers, yet firms with high prices still have considerable market share. This throws doubt on whether Bertrand is a realistic assumption for retail electricity markets.

Demand Model $\hat{\mathcal{D}}$

We will use a linear formulation similar to that found in Vives (1999), adjusted to assume that total consumer demand for electricity is inelastic and equal to X. Denote the market-share for firm f by ϕ_f. If every firm $i \in \mathcal{E}$ offers price p_i, then market share for firm $f \in \mathcal{E}$ is

$$\hat{\phi}_f = \frac{1}{|\mathcal{E}|}\left[1 - b\sum_{i\in\mathcal{E}\setminus\{f\}}(p_f - p_i)\right], \tag{2}$$

where $|\mathcal{E}|$ is the number of firms in set $\mathcal{E}$, and b reflects the willingness of consumers to switch.[8]

Given a overall demand of X, we can compute the demand for an individual firm by multiplying the firm's market share by X.

$$\hat{x}_f = X\hat{\phi}_f. \tag{3}$$

Note that in this model the overall demand is inelastic; however, for the individual firms, demand is lost or gained based on the inverse maximum switching cost, b, and the price differences between firms.

The demand model ($\hat{\mathcal{D}}$) is only valid if prices are such that all firms have positive demand. If this condition is violated, some firms will have negative demand which is incorrect and arises because $\hat{\mathcal{D}}$ does not include all the boundary conditions that are present in the actual demand model. With an arbitrary number of firms offering arbitrary prices, the true demand model is very difficult to write as a closed-form expression, thus for computational reasons (and for ease of understanding) we express the full demand model as an algorithm.

Demand Model $\mathcal{D}$

If each firm $i \in \mathcal{E}$ chooses a price p_i, we can compute demand for firm f from the following algorithm.

1. Define $\bar{\mathcal{E}}$ initially to be the set of firms who have entered the retail market i.e. $\bar{\mathcal{E}} = \mathcal{E}$.

[8] If b is small, the cross elasticity is low, meaning that firms cannot easily gain or lose customers through changing their prices, whereas if b is large, small price differences can lead to large numbers of consumers switching.

2. For each firm $f \in \bar{\mathcal{E}}$, compute $\phi_f = \frac{1}{|\bar{\mathcal{E}}|}\left[1 - b\sum_{i\in\bar{\mathcal{E}}\setminus\{f\}}(p_f - p_i)\right]$. For each firm $f \in (\mathcal{F}_g \cup \mathcal{F}_r)\setminus\bar{\mathcal{E}}$, set $\phi_f = 0$.
3. If $\phi_f \geq 0$, $\forall f \in \bar{\mathcal{E}}$ then end.
4. Remove $\arg\max_{f\in\bar{\mathcal{E}}}\{p_f\}$ from $\bar{\mathcal{E}}$.[9]
5. Go to step 2.

Below we present a theorem which states that any firm re-entering the market at the termination of the above algorithm would receive a negative market-share. This ensures that the solution computed from this algorithm is consistent with the assumptions.

THEOREM 1. *Any firm who is not in the market at the end of the above algorithm would receive a negative market-share if they were to re-enter.*

PROOF: According to the algorithm a firm will leave the market if its market-share is negative. Thus if a firm exits, it cannot immediately re-enter. Moreover, any firm that had exited previously must have a retail price higher than the most recent firm to exit.

Therefore, since the most recent firm to have exited would receive a negative demand if it were to re-enter, if any other firm were to re-enter it would also receive a negative market share (since its retail price is higher). ∎

3.2. Wholesale Market

In the retail market, firms only form contracts. No electricity is actually produced or supplied at this stage. Generation and delivery take place in the second stage of the market, typically known as the *Wholesale Market*. At the time the wholesale market is run, any uncertainty in the market (see below for more details) is resolved, so all parameters, including demand in particular, are known exactly before any offers are placed. The wholesale market is a standard nodal pricing market. Generators make offers to sell a certain quantity of electricity at a specified price. Retailers submit

[9] Note that the firm removed must have negative market-share, since ϕ_f is a decreasing function of p_f, and at least one firm has negative market-share.

how much demand they need to satisfy (since demand is inelastic, there is no point in submitting bids). The market will dispatch generators in such a way to satisfy demand at the lowest possible cost.

The focus in this paper is on the retail market, so for simplicity we assume that the wholesale market is perfectly competitive. In this we follow the example of Baldursson and von der Fehr (2007). This is the simplest assumption about the behaviour of the firms, since the offers are set to the plants' marginal costs. Given these offers, a system operator solves an optimization problem (the economic dispatch problem) to minimize the cost of satisfying the demand, yielding the optimal generation and nodal prices. These nodal prices are defined to be the marginal cost of an additional demand at a node.

3.3. Uncertainty

In this paper we consider two possible sources of uncertainty that are representative of other types. One source arises from demand uncertainty from retail consumers. In this model, the parameter X in equation (3) may be a random variable drawn from a known distribution. This uncertainty is resolved after the retail market prices are set, but before the wholesale market clears. Another possible source of risk is the cost of water to firms owning hydro generation. When water storage is low, the opportunity cost of using water is higher. However, future river flows and lake levels are unknown when prices are set in the retail market, so firms are uncertain about the value of storage. These parameters are realized before the wholesale market.

We define Ω to be the set of possible realizations of uncertainty. For a given $\omega \in \Omega$, we superscript any variables whose outcome depends upon the realization of uncertainty; e.g. x_f^ω is the demand for electricity from firm f in state ω. Any uncertainty in the model is resolved between the first and second stages. That is, firms deciding whether to enter and then setting prices in the retail market do so without knowledge of the actual realisation of the uncertain variables – they know only the distribution of the variables. However, all uncertainty is resolved before the wholesale market opens, so firms making offers at this stage do so with full information.

3.4. Profit

Each retailer in this model, vertically integrated with a generator or not, seeks to maximize (risk-adjusted) profit. Let $\mathcal{E}$ be the set of retail market entrants; for a given outcome, $\omega \in \Omega$, in the wholesale market, the profit earned by a retailer is

$$\pi_f^\omega(\mathcal{E}) = \begin{cases} (p_f - c^\omega)\, x_f^\omega + P_f^\omega, & f \in \mathcal{E}, \\ P_f^\omega, & \text{otherwise}, \end{cases}$$

where c^ω is the wholesale clearing price and P_f^ω is the profit firm f makes in the wholesale market under realisation $\omega \in \Omega$.

In the risk-neutral case, a firm will simply be interested in maximizing its expected profit in the retail plus wholesale markets. In the risk-averse case, the firm will instead maximize expected profit less a weighted measure of risk (in this case: CVaR), as we discussed in section 2. Using $\theta \in [0,1]$ as a weighting on risk and return, we define the following risk-adjusted profit function for a firm f

$$\pi_f = (1-\theta)\, E_\omega\left[\pi_f^\omega\right] - \theta \mathrm{CVaR}_\alpha\left(\pi_f^\omega\right). \tag{4}$$

In the retail market, firms simultaneously choose prices so as to maximize their risk-adjusted profit function. For a given set of firms, $\mathcal{E}$, having entered into the market, we denote the retail market game using demand model $\mathcal{D}$ as $\mathcal{G}_\mathcal{E}$ (or $\hat{\mathcal{G}}_\mathcal{E}$ when demand model $\hat{\mathcal{D}}$ is used).

Later when we introduce the full game, we will prove that under certain conditions, the equilibria arising from using the two different demand models are identical. We leave this discussion for Section 3.5.

LEMMA 1. *When firms are risk-neutral, all firms in the retail market choose identical prices for their retail sales, and each firm gets an equal market share. Furthermore, as the number of firms in the market increases, the equilibrium retail prices tend to the demand-weighted average wholesale prices.*

3.5. Full Model

With the above structure in place, we can formally define the full model. The structure is as follows. At stage 0, each firm $f \in \mathcal{F}_g \cup \mathcal{F}_r$ simultaneously decides whether to enter the retail market or not. Some set $\mathcal{E} \subseteq \mathcal{F}_g \cup \mathcal{F}_r$ choose to enter. (Note that because we assume all retail-only firms are symmetric, our results will merely give the number of such firms entering at equilibrium, not which firms.) At stage 1, retail firms choose prices as described in Section 3.1 and realise retail contract demand accordingly, and at stage 2 generators make bids in the wholesale market as described in section 3.2, which subsequently clears to realize the state-dependant wholesale price. Firms then realize profits from both the retail market and the wholesale market.

The full game comprising all three stages will be denoted $\mathcal{G}$ if demand model $\mathcal{D}$ is used (or $\hat{\mathcal{G}}$ when $\hat{\mathcal{D}}$ is used). In the appendix, we prove that, under certain conditions, the equilibria for games $\mathcal{G}$ and $\hat{\mathcal{G}}$ are identical.[10] Thus, for the remainder of this paper will restrict our interest to the game $\hat{\mathcal{G}}$.

Under the assumption that the consumers have no preference toward incumbents over new entrants we can once again use the demand functions given by equation (2); this equation is repeated below for convenience:

$$\hat{\phi}_f = \frac{1}{|\mathcal{E}|}\left[1 - b \sum_{i \in \mathcal{E}\setminus\{f\}} (p_f - p_i)\right]. \tag{5}$$

4. Example: Retail Competition at a Single Node

In this section, we will construct an example of retail competition at a single node, using the model introduced in section 3. We will first examine the outcome of the retail competition stage if all firms are risk-neutral, and then compare this to the equilibrium when one or both firms are risk-averse. Finally, we will couple the first-stage entry model with the retail competition stage and investigate how risk-aversion affects the number of firms willing to enter the retail market.

[10] See Lemma 7 and Theorem 2 for details.

4.1. Risk-neutral Case

Consider a market where the network has a single node, and suppose retail demand is equal to 150MW and totally inelastic. In the market there are two firms, each with a retail base. Firm A owns a hydro plant with capacity 100MW and cost \$$h$/MWh and firm B owns a thermal plant with capacity 100MW and cost \$50/MWh. The hydro costs are uncertain when the retail prices are chosen, however, it is common knowledge that the hydro costs, h are distributed uniformly between 0 and 100.

First, we solve for the optimal dispatch as a function of h. This gives the following dispatch quantities, q_A, q_B for each plant and clearing price, c.

$$q_A = \begin{cases} 100, \ h \le 50, \\ 50, \ \ h > 50, \end{cases}$$
$$q_B = \begin{cases} 50, \ \ h \le 50, \\ 100, \ h > 50, \end{cases}$$
$$c = \begin{cases} 50, \ h \le 50, \\ h, \ \ h > 50. \end{cases}$$

Now we can compute the expected profits for the firms as

$$\begin{aligned} \pi_A &= E\left[q_A \times (c-h) + x_A\left(p_A - c\right)\right] \\ &= \tfrac{1}{100}\left[\int_0^{50} 100\left(50-h\right)dh + x_A\int_0^{50}\left(p_A - 50\right)dh + x_A\int_{50}^{100}\left(p_A - h\right)dh\right] \\ &= 1250 + x_A\left(p_A - 62.5\right). \end{aligned}$$

Substituting in x_A (with $b = \frac{1}{75}$) and differentiating with respect to p_A gives

$$\frac{\partial \pi_A}{\partial p_A} = 75 - 2p_A + p_B + 62.5.$$

Similarly for firm B we find

$$\frac{\partial \pi_B}{\partial p_B} = 75 - 2p_B + p_A + 62.5.$$

Assuming an interior solution, these first order conditions yield equilibrium prices

$$p_A^e = p_B^e = 137.5.$$

Observe that here, in spite of the different generation assets that the firms own, they set identical prices in the retail market and share the retail demand equally. This agrees with the general result of the previous section.

4.2. Risk-Averse Case

In a risk-averse setting, the symmetry we observed in Lemma 1 is lost. This is because the outcome of the wholesale market cannot be merely treated as a constant in the risk-adjusted profit maximization problem. To examine the impact of this change, we extend the example introduced in section 4.1 to incorporate risk using the CVaR measure.

For this example, we will define the CVaR risk level to be α=0.1, and examine the effect of varying θ on the behaviour of the firms. In order to compute the conditional value at risk, we need to rank the profit for each firm as a function of h for each retail price it may choose. The profit for a given h and p_A for firm A is

$$\pi_A^h = \begin{cases} (p_A - 50)(75 - p_A + p_B) + 100(50 - h), & h \leq 50, \\ (p_A - h)(75 - p_A + p_B), & h > 50. \end{cases}$$

Note from the above equation that so long as the retail demand is positive, the profit of firm A is decreasing in h. The profit for a given h and p_B for firm B is

$$\pi_B^h = \begin{cases} (p_B - 50)(75 - p_B + p_A), & h \leq 50, \\ (p_B - h)(75 - p_B + p_A) + 100(h - 50), & h > 50. \end{cases}$$

From above we can see that firm B's profit is constant over $h \in [0, 50]$, however for $h > 50$ we have

$$\frac{\partial \pi_B^h}{\partial h} = 25 + p_B - p_A,$$

hence if $p_A - p_B \geq 25$ then the profit is decreasing in h, whereas if $p_A - p_B \leq 25$ the profit is increasing in h; it can be shown that the former case is not possible at equilibrium. Therefore here we try to find an equilibrium where $p_A - p_B \leq 25$; assuming this inequality is valid, we can write the risk-adjusted profit functions for both firms as follows

$$\begin{aligned}\pi_A &= \frac{1-\theta_A}{100}\int_0^{100} \pi_A(h)\,dh + \frac{\theta_A}{10}\int_{90}^{100} \pi_A(h)\,dh \\ &= (1-\theta_A)\left[(p_A - 62.5)(75 + p_B - p_A) + 1250\right] + \theta_A\left[(p_A - 95)(75 + p_B - p_A)\right],\end{aligned}$$

$$\begin{aligned}\pi_B &= \frac{1-\theta_B}{100}\int_0^{100} \pi_B(h)\,dh + \frac{\theta_B}{10}\int_{0}^{10} \pi_B(h)\,dh \\ &= (1-\theta_B)\left[(p_B - 62.5)(75 + p_A - p_B) + 1250\right] + \theta_B\left[(p_B - 50)(75 + p_A - p_B)\right].\end{aligned}$$

We can then solve for the equilibrium, from first order conditions, giving

$$p_A^e = \frac{5}{6}(165 + 26\theta_A - 5\theta_B), \quad p_B^e = \frac{5}{6}(165 + 13\theta_A - 10\theta_B).$$ [11]

Note that if we set $\theta_A = \theta_B = 0$ then we recover the risk-neutral equilibrium. Moreover, as firm A's risk-aversion weighting is increased the equilibrium prices of both players increase; whereas as firm B's risk-aversion weighting is increased the equilibrium prices of both players decrease. This result is due to the wholesale market positions of the firms, and the way risk-averse tends to emphasize the worst-case scenarios.

Firm A, who owns hydro generation receives the least profit when hydro costs are high, thus a risk-averse firm emphasizes these occurrences and when optimizing in the retail market sees a higher risk-adjusted 'average' spot price. Conversely, firm B, who owns thermal generation receives the least profit when hydro costs are low, thus when determining its optimal retail price sees a lower risk-adjusted 'average' spot price. Furthermore, the effects of competition mean that the risk-aversion of one firm will affect the other's optimal pricing strategy.

This example has illustrated how risk-aversion can affect the retail equilibrium prices.

[11] Note that we needed to verify this equilibrium, by considering incentives for the players to deviate such that $p_A - p_B > 25$.

4.3. Endogenous Entry

We can extend the example introduced in sections 4.1 and 4.2 to study when and how many firms would enter the market under varying conditions.

Using this demand function (with $X = 150$ and $b = \frac{1}{75}$), we can solve for the equilibrium retail prices as a function of the number of firms entering the market, for various levels of risk-aversion (θ). For the purposes of this example we will assume that the two firms owning generation assets will always participate in the retail market. In the table below we show the equilibrium retail prices for for different numbers of retailers n (≥ 2).

n	Retailer (Hydro)	Retailer (Thermal)	Retailers (No generation)
2	\$137.50	\$137.50	N/A
3	\$100.00	\$100.00	\$100.00
4	\$87.50	\$87.50	\$87.50
5	\$81.25	\$81.25	\$81.25
6	\$77.50	\$77.50	\$77.50

Table 1 Risk-Neutral ($\theta = 0.0$)

n	Retailer (Hydro)	Retailer (Thermal)	Retailers (No generation)
2	\$138.75	\$123.75	N/A
3	\$107.25	\$89.25	\$107.25
4	\$97.32	\$78.04	\$97.32
5	\$92.50	\$72.50	\$92.50
6	\$89.66	\$69.20	\$89.66

Table 2 Risk-Averse ($\theta = 0.5$)

n	Retailer (Hydro)	Retailer (Thermal)	Retailers (No generation)
2	\$140.00	\$110.00	N/A
3	\$114.50	\$78.50	\$114.50
4	\$107.14	\$68.57	\$107.14
5	\$103.75	\$63.75	\$103.75
6	\$101.82	\$60.91	\$101.82

Table 3 Risk-Averse ($\theta = 1.0$)

It can be shown that as the number of entrants into the retail market increases the equilibrium prices for the firms tend towards the wholesale price when firms are risk neutral, however, risk-aversion leads to the retailer with thermal generation to reduce its retail price, and all the other retails to increase their prices.

n	Retailer (Hydro)	Retailer (Thermal)	Retailers (No generation)
∞	$\$62.50 + 32.50\theta$	$\$62.50 - 12.50\theta$	$\$62.50 + 32.50\theta$

Table 4 Equilibrium prices as the number of firms becomes large.

In the following graph we plot the risk-adjusted profits of the entering firms (those without generation assets) as functions of risk-aversion parameter θ. In this plot we have included a *cost of entry*; this horizontal line is the line above which firms would have had incentive to enter the market. When the profit is below this line one firm (or more) would have not entered the market at equilibrium.

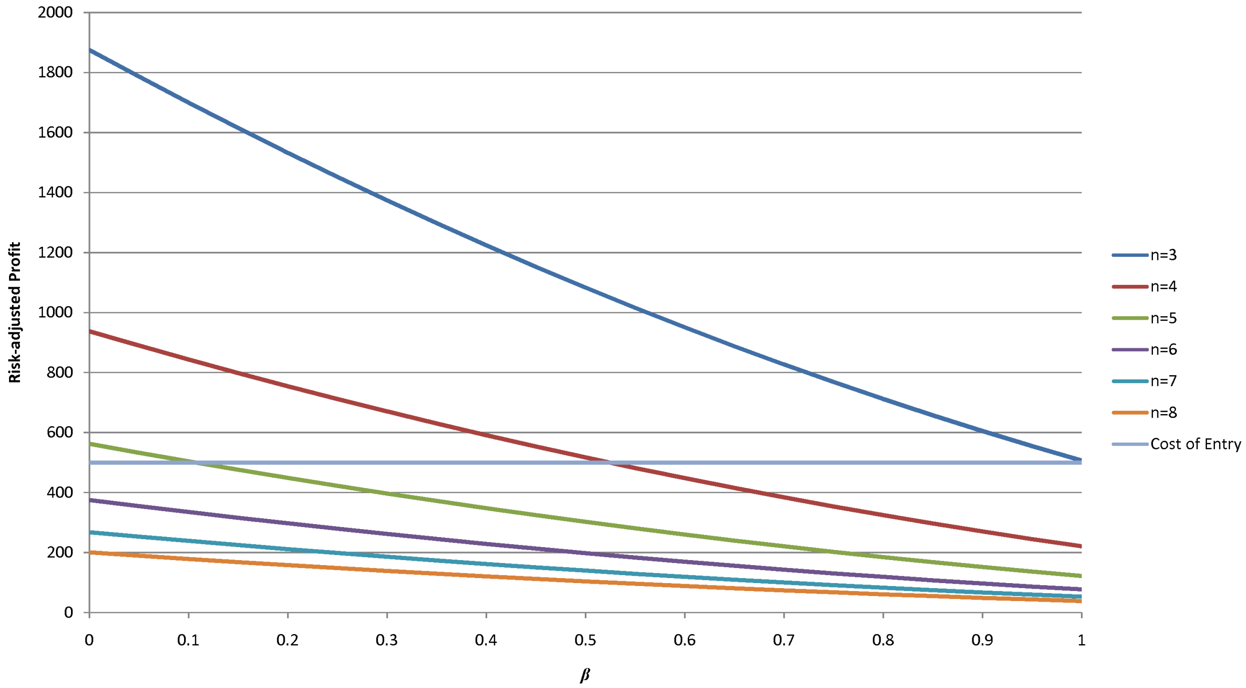

Figure 1 Risk-adjusted profit for retailers for varying numbers of entrants.

We can therefore see that for a cost of entry of 500 [dollars per period?], for $\theta \in [0, 0.107]$ three retailers enter the market, for $\theta \in (0.107, 0.524]$ two retailers enter, and finally for $\theta \in (0.524, 1]$ only one retail-only firm will enter the market, along with the two firms owning generation. The expected profits and the equilbrium prices for all firms as a function of risk-aversion (with the equilibrium number of firms entering the market) can be seen in figures 2 and 3.

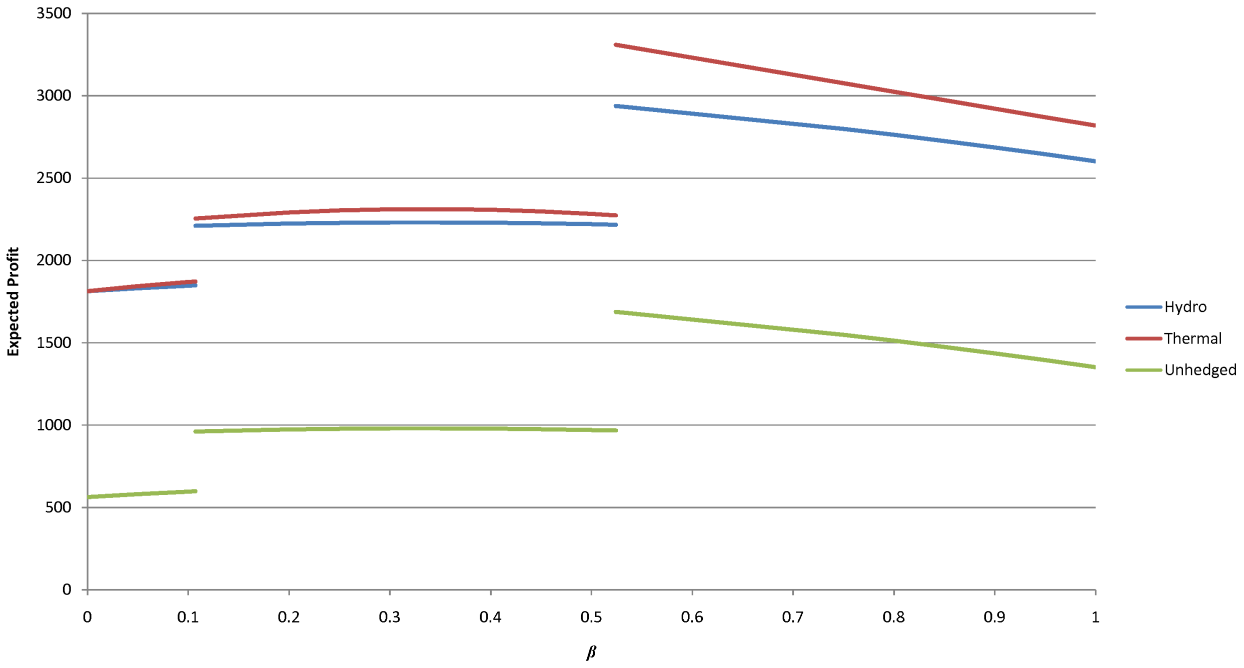

Figure 2 Expected profit for retailers as function of risk-aversion with optimal number of entrants.

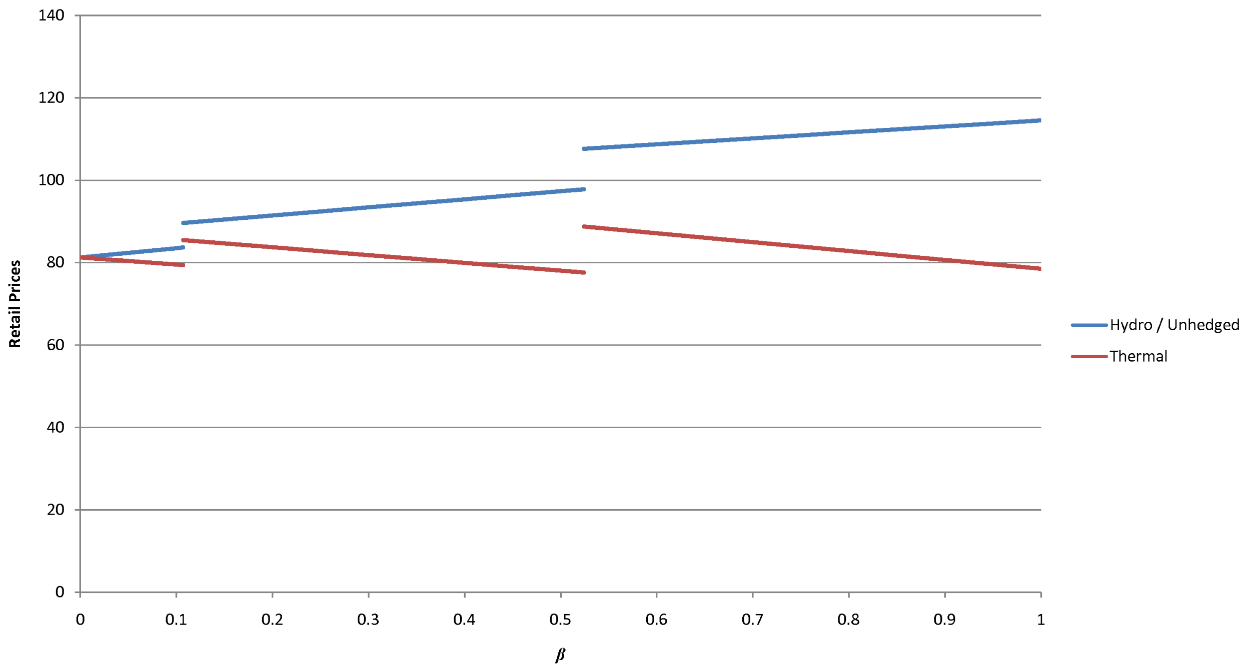

Figure 3 Equilibrium retail prices as function of risk-aversion with optimal number of entrants.

We can see that as firms become more risk-averse, fewer have incentive to enter the retail market, allowing the firms already in the market to charge higher prices and therefore make higher expected profits; this, however, is somewhat offset by the risk-aversion causing sub-optimal (in expectation) pricing decisions. Moreover, for low levels of risk-aversion, three unhedged retailers enter the market; these firms have incentive to increase their retail pricing as they become more risk-averse (as they are concerned with the possibility of high wholesale prices) this aversion leads to higher expected profits for retailers. However, as the risk-aversion increases, fewer of these firms enter the market, and the thermal owner, who benefits from higher prices, becomes more dominant. This leads to a situation where all firms' profits decrease as they become more risk-averse.[12]

4.4. Summary

So far in the paper we have introduced a three-stage model of risk-averse retailers competing in an electricity market. In the single-node examples that we have examined the uncertainty has been represented by through continuous distributions and the examples have been solved analytically. In the next section we will consider multiple-node transmission networks, and use monte-carlo sampling to create the possible scenarios. We will solve these problems numerically as complementarity problems.

5. Electricity Market Model with Multiple Nodes

Finally we extend the model introduced in section 3 to accommodate networks. Assume the market now operates on a network with multiple nodes, $n \in \mathcal{N}$. We denote by $\mathcal{E}_n \subseteq \mathcal{F}_g \cup \mathcal{F}_r$ the set of firms that have entered the retail market at node n. As before, there is uncertainty about future demand levels and nodal prices. The different scenarios are indexed by $\omega \in \Omega$.

In order to find the most efficient way to allocate electricity over a network, a dispatch model is required. This means that it is more difficult to model the uncertainty using continuous distribution functions. Thus we turn to discrete distribution functions instead, and employ computational methods to find equilibria of the model. We incorporate CVaR using a formulation from Choi and Ruszcynski (2008).

[12] The thermal generator's expected profit decreases because it is charging a lower retail price than it would if it were risk-neutral, whereas the other firms' profits decrease because their customer base is being eroded by the thermal generator's low retail prices.

The notation for this section extends the model of section 3 with the addition of a subscript n where a parameter is specific to a node n. Where a parameter is also refers to a particular firm, we adopt the convention that the node subscript comes first, followed by the firm subscript. The parameters and variables are ammended as listed below.

Parameters:

- c_n^ω is the wholesale price in scenario ω at node n;
- X_n^ω is the total retail demand in scenario ω at node n;
- p_{ni} is the retail price offered by firm i at node n, the matrix p_{-f} contains all these values other than those pertaining to firm f;
- P_f^ω is the wholesale profit for firm f in scenario ω;
- ρ is the probability of scenario ω.

Variables:

- η_f is the α-quantile profit of firm f;
- v_f^ω is the positive deviation from the α-quantile profit of firm j;
- w_f^ω is the negative deviation from the α-quantile profit of firm j;
- ϕ_{nf} is the realised retail market-share for firm f at node n;
- π_f^ω is the profit for firm f in scenario ω;
- p_{nf} is the retail price offered by firm f at node n.

Below we give a mathematical program that maximizes a single firm's risk-adjusted expected profit, given the retail prices chosen by the other firms (assuming demand model $\hat{\mathcal{D}}$).

$$
\begin{aligned}
\Pi_f\left(p_{-f}\right) = \max \ & \sum_{\omega\in\Omega} \rho\pi_f^\omega - \tfrac{\theta}{\alpha} \sum_{\omega\in\Omega} \rho\left[(1-\alpha)\, w_f^\omega + \alpha v_f^\omega\right] & & \\
\text{s.t. } & \pi_f^\omega = \eta_f + v_f^\omega - w_f^\omega & \left[\lambda_f^\omega\right] & \ \forall \omega\in\Omega \\
& \pi_f^\omega = \sum_{n\in\mathcal{N}} \left[X_n^\omega \sum_{i\in\mathcal{E}_n\cap\{f\}} \left(p_{ni} - c_n^\omega\right) \hat{\phi}_{ni}\right] + P_f^\omega & \left[\mu_f^\omega\right] & \ \forall \omega\in\Omega \\
& \hat{\phi}_{nf} = \tfrac{1}{|\mathcal{E}_n|}\left[1 + b \sum_{i\in\mathcal{E}_n\setminus\{f\}} \left(p_{ni} - p_{nf}\right)\right] & \left[\nu_{nf}\right] & \ \forall n\in\mathcal{N} \\
& w_f^\omega, v_f^\omega \geq 0 & & \ \forall \omega\in\Omega.
\end{aligned}
$$

The above problem, $\mathcal{P}_f\left(p_{-f}\right)$, can be shown to be equivalent to a convex optimization problem (see Lemma 2 in the appendix). Therefore the solution to $\Pi_f(P_{-f})$ is the solution to the system of KKTs for $\Pi_f(P_{-f})$, given below (and these are valid for firm $f \in \mathcal{E}$).

$$\begin{aligned}
\pi_f^\omega &= \eta_f + v_f^\omega - w_f^\omega \quad \forall \omega \in \Omega, \forall f \in \mathcal{F} \\
\sum_{n\in\mathcal{N}} \left[X_n^\omega \sum_{i\in\mathcal{E}_n \cap \{f\}} (p_{ni} - c_n^\omega)\,\hat{\phi}_{ni} \right] + P_f^\omega - \pi_f^\omega &= 0 \quad \forall \omega \in \Omega, \forall f \in \mathcal{F} \\
\hat{\phi}_{nf} = \frac{1}{|\mathcal{E}_n|} &\left[1 + b \sum_{i\in\mathcal{E}_n \setminus \{f\}} (p_{ni} - p_{nf}) \right] \quad \forall f \in \mathcal{E}_n, \forall n \in \mathcal{N} \\
-\rho + \lambda_f^\omega + \mu_f^\omega &= 0 \quad \forall \omega \in \Omega, \forall f \in \mathcal{F} \\
-\sum_{\omega\in\Omega} \lambda_f^\omega &= 0 \quad \forall f \in \mathcal{F} \\
-\sum_{\omega\in\Omega} X^\omega \hat{\phi}_{nf} \mu_f^\omega + b\frac{|\mathcal{E}_n| - 1}{|\mathcal{E}_n|} \nu_{nf} &= 0 \quad \forall f \in \mathcal{E}_n, \forall n \in \mathcal{N} \\
\nu_{nf} - \sum_{\omega\in\Omega} \left[X_n^\omega \sum_{i\in\mathcal{E}_n \cap \{f\}} (p_{ni} - c_n^\omega)\mu_f^\omega \right] &= 0 \quad \forall f \in \mathcal{E}_n, \forall n \in \mathcal{N} \\
0 \le v_f^\omega \perp \theta\rho - \lambda_f^\omega &\ge 0 \quad \forall \omega \in \Omega, \forall f \in \mathcal{F} \\
0 \le w_f^\omega \perp \theta\frac{1-\alpha}{\alpha}\rho + \lambda_f^\omega &\ge 0 \quad \forall \omega \in \Omega, \forall f \in \mathcal{F}.
\end{aligned}$$

This system of equations is known as a mixed complementarity problem (MCP), which can be solved in `GAMS` using the `PATH` solver (2000). Since each firm is solving a convex optimization problem, any solution to the above MCP gives an equilibrium for $\hat{\mathcal{G}}_{\mathcal{E}}$.

In the appendix we will prove that the equilibrium solution we find from the MCP above, is indeed an equilibrium to the game using the demand model $\mathcal{D}$....EXPAND?***

6. Application to Asset Swapping

In October 2010, the New Zealand Government reorganized the New Zealand Electricity Market. One of the changes implemented was an ‘asset transfer’, in which one of the state-owned firms purchased generation assets from another; this sale took place in June 2011. One of the key aims of this asset transfer was to improve competition in the *retail* electricity market. All major firms in New Zealand are vertically integrated, owning both generation assets and a portfolio of retail consumers. Despite the existence of five large retailers nationally, many areas of the country are served by no more than two retailers, a fact believed to be due to nodal price risk in the wholesale market. The goal of the asset swap was to give more of the vertically integrated firms a geographically diverse portfolio of generation assets, thus giving those firms the confidence to enter into various retail markets around the country.[13]

Nodal price risk is believed to be a serious problem in the New Zealand market. New Zealand is dominated by hydro generation, however, there is limited water storage, so the availability of hydro generation is dependent on the weather. In so called 'dry years', river inflows fall and hydro plants must cut back on generation to conserve water. The resulting constriction of supply causes wholesale prices to rise up to 1000% above average. Firms who rely on hydro generation to satisfy their retail contractions must then buy expensive electricity on the wholesale market and sell at loss in their retail market (since the latter is contractually fixed). This is one source of risk. The other source of risk is due to geographical location. New Zealand has two main islands. The South Island contains excess generation, the North Island conversely has most of the load. If the inter-island cable fails, prices will then diverge sharply between the two islands. Thus a firm without assets in one island will be unlikely to build a retail base in that island, as they will be exposed to wholesale price fluctuations when the inter-island cable is out of service. The asset swap implemented in 2010 is expected to reduce both sources of risk, but formal analysis has been limited to the effect on the wholesale market (Downward et al., 2011).

We can use the model presented in this paper to analyze the impact of such an asset swap on the *retail* market. In particular, we can look at such questions as whether an asset swap causes more retail competition, including new entrants to each market (North and South). We investigate two types of asset swap. One is a physical asset swap, as described above, where two firms swap ownership of physical assets. The other is a *virtual* asset swap, where the two firms enter a long-term contract to 'swap' electricity at two nodes. The virtual asset swap was another recommendation from the 2009 Electricity Technical Advisory Group.

[13] See the discussion document from the 2009 Electricity Technical Advisory Group (2009) for more background on the asset swap. This report stated that this reallocation should improve competition in the retail market, by incentivising *gentailers* (firms with both generation and retail arms) to enter into new markets; thereby driving prices down for consumers.

6.1. Example: Risk-averse Firms with Fixed Entry

In this example, we will present a two-node network (North and South) with three firms (A,B and C). Prior to any swapping of assets, firm A has two thermal plants in the North, firm B has has one thermal plant in the North and one hydro plant in the South and Firm C two hydro plants in the South. The asset swap that we consider will consist of firms A and C swapping a thermal in the North and a hydro in the South. These allocations are depicted in figure 4 below. We will also compare this physical swap with a virutal swap, where a financial contract is established, which trades North power for South power between firms A and C. Ignoring any fixed payments, this contract, for x MW of power, gives the following payments to the firms (or costs, if negative). For firm A the payment is: $(p_S - p_N)\,x$; and for firm C the payment is $(p_N - p_S)\,x$, where p_N and p_S are the spot prices for North and South power respectively.

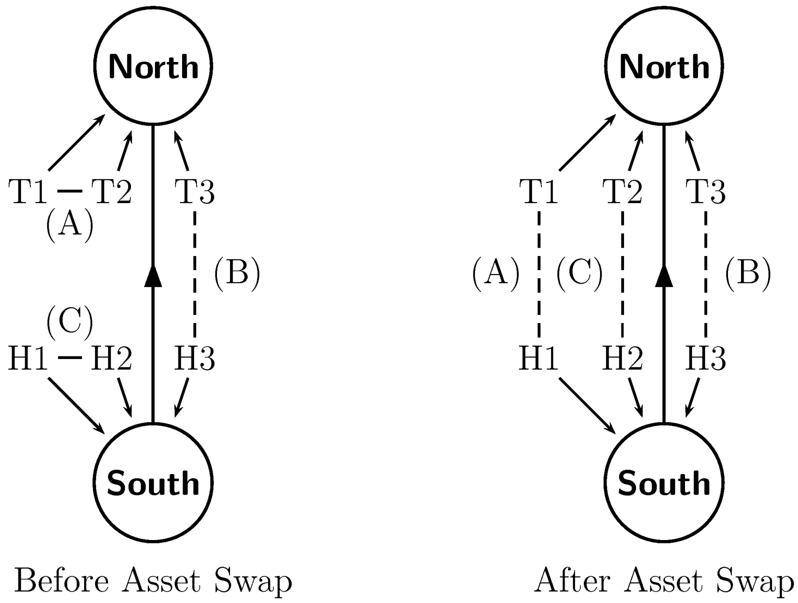

Figure 4 Physical asset swap.

The production cost of each of the thermal plants is

$$c_T\,(q) = 50q + 0.05q^2,$$

and for the hydro it is

$$c_H(q) = 10q + hq^2,$$

where h is a random variable representing water scarcity. For the upcoming examples, h is normally distributed, with mean 0.3 and standard deviation 0.1.

The demand in the North is $d_N = 2000$ MW, the demand in the South is $d_S = 1000$ MW, and the nominal capacity of the HVDC is 250 MW, with a 5% chance of an outage (note that the probability of an outage is independent of h). Given these parameters, using Monte Carlo simulation, we solve the dispatch problem (assuming competitive offering behaviour) and record the nodal prices and firm profits. The distributions of nodal price differences and firm profits are shown in figures 5 and 6 below.

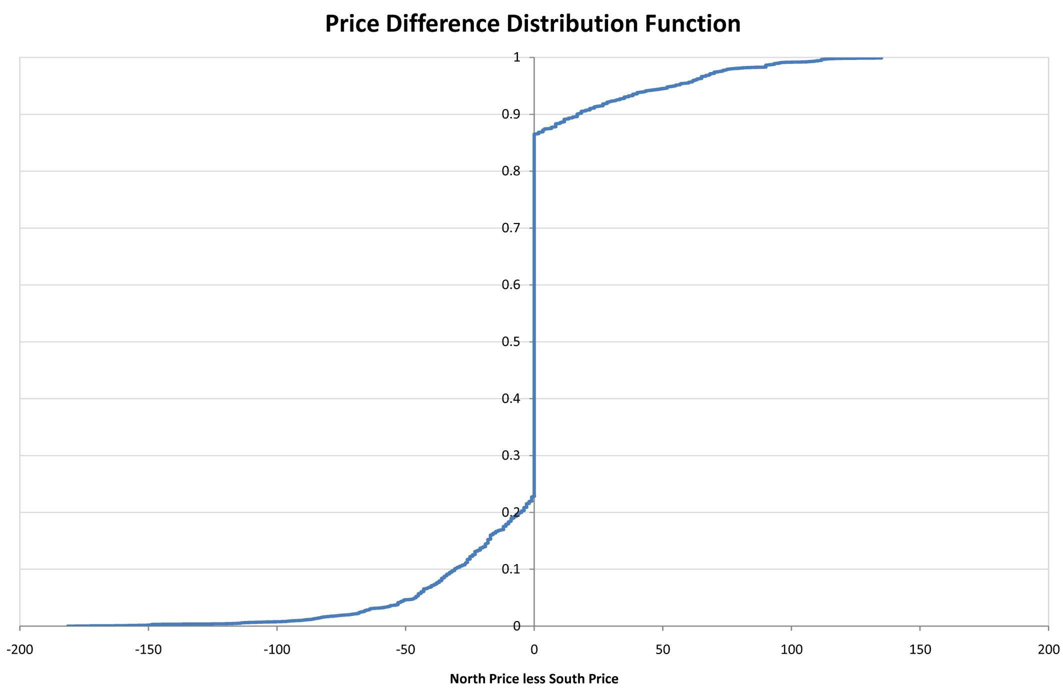

Figure 5 Cumulative distribution function for price difference.

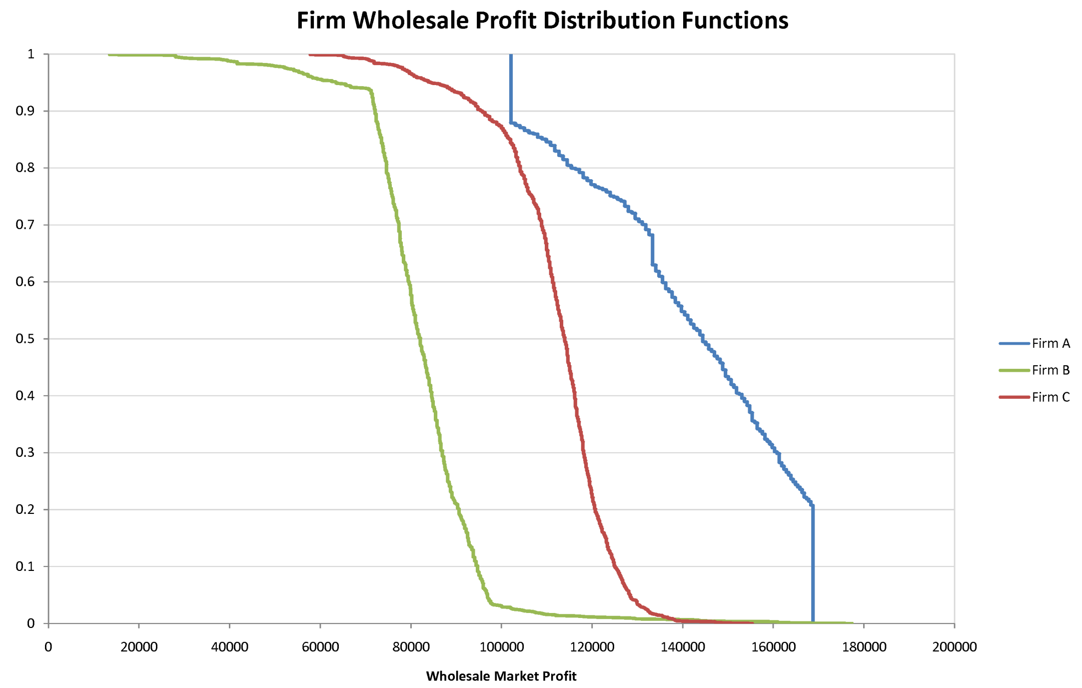

Figure 6 Cumulative distribution function for wholesale profits.

Given which players are in the market, using the formulation detailed in section 5, we can compute the equilibrium prices for the three firms. For simplicity, we will assume that firm A and B are initially in the retail market in the North and firms B and C are in the retail market in the South and that none of these firms wish to exit (one could think of this as the firms having a high exit cost).

Equilibrium computations here are made numerically (due to the dispatch problem), thus we need to set the parameters specifying the risk-aversion parameters and the cross-elasticity of the demand. We set the risk-aversion parameters for all three firms to be identical: $\alpha = 0.1$, and $\theta = 0.5$ – this means that the firms are maximizing, the expected profit and the conditional expectation of the lowest 10% of profits with equal weightings. The cross-elasticity of the demand at both nodes is $b = \frac{1}{200}$.

	Genesis		Contact		Meridian		Consumer
	North	South	North	South	North	South	Costs
Before Swap	281.91	–	287.48	325.16	–	325.16	894,238
After Swap	283.58	–	288.31	325.74	–	326.33	897,702
Virtual (250MW)	281.91	–	287.48	325.16	–	325.16	894,238
Virtual (500MW)	281.91	–	287.48	325.16	–	325.16	894,238

Table 5 Status Quo Retail Entrants

We are interested in determining how the equilibrium pricing changes once a physical or virtual asset swap takes place. To do this we first compute the equilibrium pricing under various swaps assuming the entry is the same as the *status quo* described above. In table 5 we present the equilibrium pricing before and after the physical asset swap, as well as two virtual swaps. This shows that if the physical asset swap is carried out and no new entrants enter the market then the overall costs to the retail customers increases, whereas with the virtual swaps there is no change to the equilibrium retail pricing. As the firms are less well hedged (i.e. they have less generation where their customers are located), their risk-aversion entices them to increase their prices thereby reducing their customer base.

Now we will consider the situation where there is full entry (i.e. all three firms are offering retail contracts at both nodes). In table 6 we show the equilibrium prices and consumer costs for the same four cases, with the **before swap** case serving as a counter-factual for comparative purposes. Once again we see that the physical asset swap leads to higher overall retail costs, however this time the virtual swaps have decreased prices and lead to lower overall costs for retail consumers. The reasons for these effects are subtle. In the case of the physical swap, since all three firms have a single thermal unit in the North and single hydro in the South none of the firms are particularly well hedged. The prices have increased because before the swap, there was one well hedged firm at each node, and this firm could offer a lower price than the others leading to higher competition and lower equilibrium pricing. The situation with the virtual swap is different, there is no physical swap, however there is a contract traded between the firms. Prices in the South drop, because Firm A effectively receives power in the South without bearing the physical risk associated with high water values.

	Genesis		Contact		Meridian		Consumer
	North	South	North	South	North	South	Costs
Before Swap	220.99	242.87	224.58	250.61	224.58	242.57	691,234
After Swap	226.38	226.38	226.38	226.38	226.38	226.38	711,248
Virtual (250MW)	218.25	236.14	223.67	251.04	223.67	251.04	686,987
Virtual (500MW)	216.48	226.13	222.82	246.14	223.02	247.31	676,326

Table 6 Full Entry

6.2. Example: Risk-averse Firms with Endogenious Entry

Finally, we consider the full game. In the first stage firms A and B decide whether or not to enter the retail market at the South and North nodes, respectively. Once the entrants are fixed, the equilibrium retail prices can be computed.

In figure 7 below, we plot the equilibrium cost to retail customers as a function of the cost of entry into the retail markets. On the right side of the figure the cost of entry is large, here under all four scenarios there is no (additional) entry into the retail market. This case was shown in table 5 above. On the left side of the figure the cost of entry is small, here under all for scenarios there is entry from both firms A and C into the South and North, respectively, as shown in table 6. For the medium entry costs there is a third case, where Firm A enters the South, but firm C does not enter the North.

From figure 7 we can see that the physical asset swap leads to higher consumer costs for much of the range of entry costs. The only times it reduces the overall costs is when the swap entices entry into the market by firm C who, prior to the swap would have preferred to have not entered (cost of entry: \$17,200 – \$23,000).

The virtual swaps not only reduce consumer costs for a particular set of entrants, it also leads to firm A entering into the South, even with large entry costs (although there is little change to the incentives for firm C to enter the North).

To summarise, if the cost of entry is large, the best way to entice entry into the market is by way of a virtual swap. Virtual assets are also the best way to reduce prices when cost of entry is low. The only time the physical swap improves welfare is to entice firm C to enter the retail market in the North, when there is a medium cost of entry.

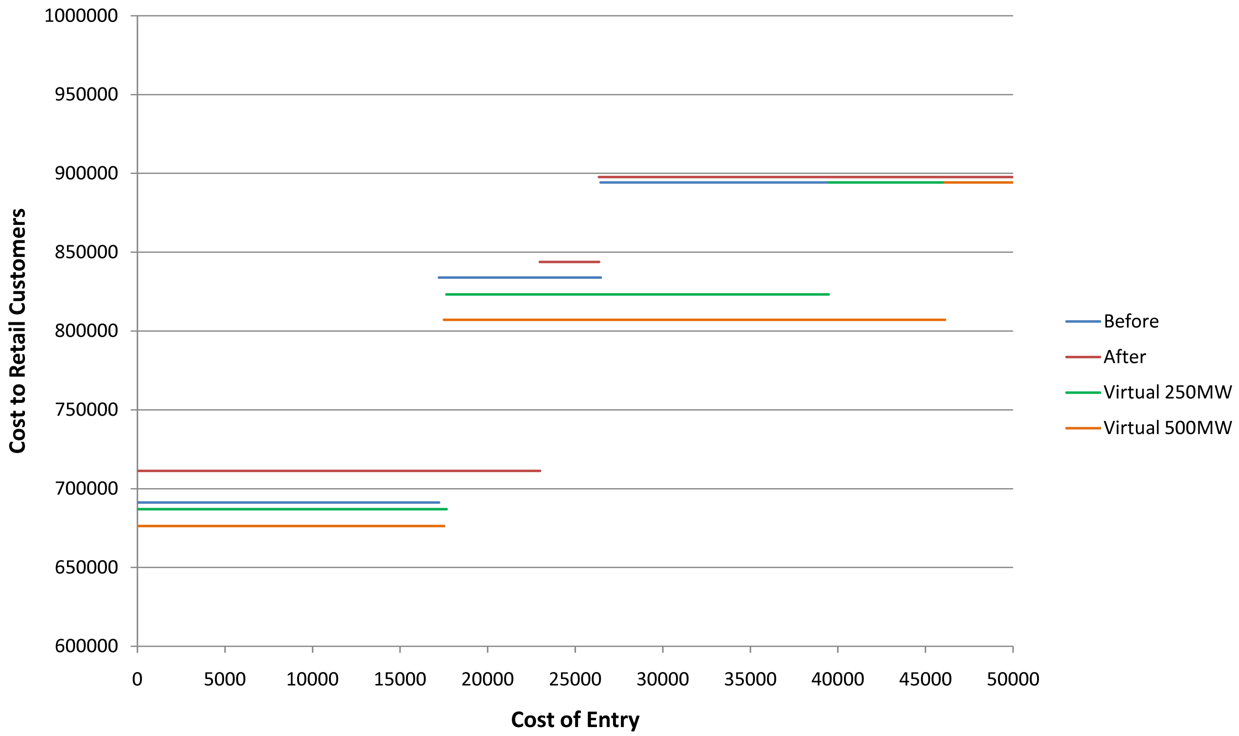

Figure 7 Comparison of retail customer costs, before and after physical and virtual asset swaps over a range of entry costs.

7. Conclusions

Motivated by Prof. Frank Wolak's report on the NZEM and the Ministerial Review into the electricity industry, we have constructed a model which encompasses both the retail and wholesale markets. We first observe that with risk-neutral firms and a competitive wholesale market, retailers will reach a symmetric equilibrium. We then examine how their behaviour may change using a simple one-node example with two firms. We find that risk-aversion does not always mean that a premium is passed onto consumers, in fact, retail prices may drop as a firm becomes more risk-averse (this effect is dependent upon the particular circumstances of the retailers).

8. Proofs

Proof of Lemma PROOF: Risk-neutral firms aim to maximize their expected profit. Given a set of firms in the retail market $\mathcal{E}$, the expected profit function for firm $f \in \mathcal{E}$ is given by

$$\pi_f = E_\omega \left[(p_f - c^\omega) \frac{X^\omega}{|\mathcal{E}|} \left(1 - b \sum_{i \in \mathcal{E} \setminus \{f\}} (p_f - p_i) \right) + P_f^\omega \right].$$

This corresponds to equation 4 where $\theta = 0$. Recall that P_f^ω is firm f's profit from the wholesale market and c^ω is the clearing price in the wholesale market, for the realization of uncertainty $\omega \in \Omega$.

Now we wish to compute the Nash equilibrium for the retail game[14]; that is a point where no firm can alter its strategy to unilaterally improve its expected profit. Since each player has a smooth, concave profit function, we can find the maximum profit from the first order condition. Differentiating the expected profit function with respect to p_f, we get

$$\frac{\partial \pi_f}{\partial p_f} = \frac{1}{|\mathcal{E}|} E_\omega [X^\omega] + \frac{b}{|\mathcal{E}|} E_\omega [X^\omega] \sum_{i \in \mathcal{E} \setminus \{f\}} p_i - \frac{b(|\mathcal{E}| - 1)}{|\mathcal{E}|} (2 p_f E_\omega [X^\omega] - E_\omega [c^\omega X^\omega]).$$

Solving for price p_f^* yields

$$\left. \frac{\partial \pi_f}{\partial p_f} \right|_{p_f = p_f^*} = 0 \Rightarrow p_f^* = \frac{1 + b \sum_{i \in \mathcal{E} \setminus \{f\}} p_i + b(|\mathcal{E}| - 1) \frac{E_\omega [c^\omega X^\omega]}{E_\omega [X^\omega]}}{2b(|\mathcal{E}| - 1)}. \tag{6}$$

At the equilibrium point, the above equation must be satisfied for all firms simultaneously. To solve for this point we first sum the above equation for all $f \in \mathcal{E}$, yielding

$$\sum_{f \in \mathcal{E}} p_f^* = \frac{|\mathcal{E}| + b(|\mathcal{E}| - 1) \sum_{f \in \mathcal{E}} p_f^* + b|\mathcal{E}|(|\mathcal{E}| - 1) \frac{E_\omega [c^\omega X^\omega]}{E_\omega [X^\omega]}}{2b(|\mathcal{E}| - 1)},$$

which can be rearranged to give

$$\sum_{f \in \mathcal{E}} p_f^* = \frac{|\mathcal{E}|}{b(|\mathcal{E}| - 1)} + |\mathcal{E}| \frac{E_\omega [c^\omega X^\omega]}{E_\omega [X^\omega]}.$$

[14] We will demonstrate that there is only one equilibrium as part of the construction.

Finally we substitute the above expression into equation (6) to find the equilibrium price for firm f.

$$p_f^e = \frac{1}{b\left(|\mathcal{E}|-1\right)} + \frac{E_\omega\left[c^\omega X^\omega\right]}{E_\omega\left[X^\omega\right]}.$$

For scenario ω the retail demand that each firm f gets is

$$x_f^* = \frac{X^\omega}{|\mathcal{E}|}.$$

In other words, we find that the firms all choose identical prices for their retail sales, and each firm gets an equal market share. This is because the profit from the wholesale market is not affected by the decisions made in the retail market, and is merely a (uncertain) constant term in the profit function. Therefore, all firms entering the retail market act symmetrically, even though some may own profitable generation. In a risk neutral setting, the expected value of this wholesale profit can be separated from the retail profit, so it plays no part in the optimization problem. On the surface this may seem peculiar, but optimizing expected profits cannot distinguish between opportunity costs and actual costs, so all firms have the same incentives. We illustrate this by way of an example in the next section.

Also note that as the number of firms in the market increases, the equilibrium retail prices tend toward demand-weighted average wholesale prices (since we have a competitive wholesale market, these prices are based on generation costs). This can be observed from the following limit

$$\lim_{n\to\infty} p_f = \lim_{|\mathcal{E}|\to\infty}\left[\frac{1}{b\left(|\mathcal{E}|-1\right)} + \frac{E_\omega\left[c^\omega X^\omega\right]}{E_\omega\left[X^\omega\right]}\right] = \frac{E_\omega\left[c^\omega X^\omega\right]}{E_\omega\left[X^\omega\right]}.$$

∎

LEMMA 2. *The optimal solution (both primal and dual) to $\mathcal{P}_f\left(p_{-f}\right)$ is equivalent to solution to the following convex mathematical program, for $\theta \in [0,1)$.*

$$\begin{aligned}
\hat{\Pi}_f(p_{-f}) = \max \ & \sum_{\omega\in\Omega} \rho\pi_f^\omega - \tfrac{\theta}{\alpha}\sum_{\omega\in\Omega}\rho\left[(1-\alpha)w_f^\omega + \alpha v_f^\omega\right] \\
s.t.\ & \pi_f^\omega = \eta_f + v_f^\omega - w_f^\omega && \left[\lambda_f^\omega\right]\ \forall\omega\in\Omega \\
& \pi_f^\omega \le \sum_{n\in\mathcal{N}}\left[X_n^\omega \sum_{i\in\mathcal{E}_n\cap\{f\}}(p_{ni}-c_n^\omega)\hat{\phi}_{ni}\right] + P_f^\omega && \left[\mu_f^\omega\right]\ \forall\omega\in\Omega \\
& \hat{\phi}_{nf} = \tfrac{1}{|\mathcal{E}_n|}\left[1 + b\sum_{i\in\mathcal{E}_n\setminus\{f\}}(p_{ni}-p_{nf})\right] && [\nu_{nf}]\ \forall n\in\mathcal{N} \\
& w_f^\omega, v_f^\omega \ge 0 && \forall\omega\in\Omega.
\end{aligned}$$

Proof: First, observe that the mathematical program presented in the statement of the lemma is a convex relaxation of $\mathcal{P}_f(p_{-f})$. The only non-linearity in the formulation is in the inequality constraining the profit, and, with a simple substitution for $\hat{\phi}$ this can be seen to be a convex quadratic.

Now we will present the KKT conditions of the above problem.

$$\begin{aligned}
\pi_f^\omega = \eta_f + v_f^\omega - w_f^\omega \quad & \forall\omega\in\Omega, \\
0 \le \mu_f^\omega \perp \sum_{n\in\mathcal{N}}\left[X_n^\omega \sum_{i\in\mathcal{E}_n\cap\{f\}}(p_{ni}-c_n^\omega)\hat{\phi}_{ni}\right] + P_f^\omega - \pi_f^\omega \ge 0 \quad & \forall\omega\in\Omega, \\
\hat{\phi}_{nf} = \frac{1}{|\mathcal{E}_n|}\left[1 + b\sum_{i\in\mathcal{E}_n\setminus\{f\}}(p_{ni}-p_{nf})\right] \quad & \forall n\in\mathcal{N} \\
-\rho + \lambda_f^\omega + \mu_f^\omega = 0 \quad & \forall\omega\in\Omega, \\
-\sum_{\omega\in\Omega}\lambda_f^\omega = 0 \quad & \\
-\sum_{\omega\in\Omega} X^\omega\hat{\phi}_{nf}\mu_f^\omega + b\frac{|\mathcal{E}_n|-1}{|\mathcal{E}_n|}\nu_{nf} = 0 \quad & \forall n\in\mathcal{N} \\
\nu_{nf} - \sum_{\omega\in\Omega}\left[X_n^\omega \sum_{i\in\mathcal{E}_n\cap\{f\}}(p_{ni}-c_n^\omega)\mu_f^\omega\right] = 0 \quad & \forall n\in\mathcal{N} \\
0 \le v_f^\omega \perp \theta\rho - \lambda_f^\omega \ge 0 \quad & \forall\omega\in\Omega, \\
0 \le w_f^\omega \perp \theta\frac{1-\alpha}{\alpha}\rho + \lambda_f^\omega \ge 0 \quad & \forall\omega\in\Omega.
\end{aligned}$$

To prove the statement of the lemma we will simply show that μ_f^ω is strictly positive at the optimal solution, and hence from the orthogonality constraint the inequality constraint must be binding. Hence, since this problem is a relaxation of the actual problem the optimal solutions must coincide. Consider the following two constraints from the KKT conditions, above:

$$\theta\rho - \lambda_f^\omega \ge 0, \quad \forall\omega\in\Omega,$$

$$-\rho + \lambda_f^\omega + \mu_f^\omega, \quad \forall \omega \in \Omega.$$

Eliminating λ_f^ω gives:

$$(\theta - 1)\,\rho + \mu_f^\omega \geq 0.$$

Since ρ is strictly positive and $(\theta - 1)$ is strictly negative, μ_f^ω must be strictly positive. This ensures that the profit inequality constraint is binding and thus the optimal solution for $\mathcal{P}_f\,(p_{-f})$ is identical to the relaxation. ∎

The following lemmas demonstrate that under certain conditions the equilibrium for $\hat{\mathcal{G}}_{\mathcal{E}}$ is also an equilibrium for $\mathcal{G}_{\mathcal{E}}$, and vice versa. In order to show this we first must prove that the demand model $\mathcal{D}$ yields continuous concave market shares. Then we can show that if there is no incentive to deviate from an equilibrium for one demand function, nor is there incentive to deviate for the other demand function.

LEMMA 3. *Suppose that we have a vector of retail prices p, and for a firm $f \in \bar{\mathcal{E}}$, under demand model $\mathcal{D}$, we obtain $\phi_f = 0$. If firm f is removed from $\bar{\mathcal{E}}$, then ϕ_j remains unchanged for all $j \in \bar{\mathcal{E}} \setminus \{f\}$.*

PROOF: First, we know that the demand of firm f is 0, i.e. $\phi_f = 0$, therefore

$$\begin{aligned}
&\frac{1}{|\bar{\mathcal{E}}|}\left[1 - b \sum_{i \in \bar{\mathcal{E}} \setminus \{f\}} (p_f - p_i)\right] = 0 \\
\Leftrightarrow &\sum_{i \in \bar{\mathcal{E}} \setminus \{f\}} (p_f - p_i) = \frac{1}{b} \\
\Leftrightarrow &\left|\bar{\mathcal{E}} - 1\right| p_f - \sum_{i \in \bar{\mathcal{E}} \setminus \{f\}} p_i = \frac{1}{b} \\
\Leftrightarrow &\sum_{i \in \bar{\mathcal{E}} \setminus \{f\}} p_i = \left(|\bar{\mathcal{E}}| - 1\right) p_f - \frac{1}{b} && (7) \\
\Leftrightarrow &\sum_{i \in \bar{\mathcal{E}}} p_i = |\bar{\mathcal{E}}| p_f - \frac{1}{b} && (8)
\end{aligned}$$

Without loss of generality we will consider the demand of firm $j \in \mathcal{E} \setminus \{f\}$, at the point where firm f is removed from the market. With firm f in the market, we have:

$$\phi_j = \frac{1}{|\bar{\mathcal{E}}|}\left[1 - b \sum_{i \in \bar{\mathcal{E}} \setminus \{j\}} (p_j - p_i)\right],$$

which can be rewritten as:

$$\phi_j = \frac{1}{|\bar{\mathcal{E}}|}\left[1 - b\left(|\bar{\mathcal{E}}|\, p_j - \sum_{i\in\bar{\mathcal{E}}} p_i\right)\right].$$

Substituting equation (8) into the above expression gives:

$$\phi_j = -b\left(p_j - p_f\right).$$

Whereas, without firm f in the market, we have:

$$\phi_j = \frac{1}{|\bar{\mathcal{E}}| - 1}\left[1 - b \sum_{i\in\bar{\mathcal{E}}\setminus\{j\cup f\}} \left(p_j - p_i\right)\right],$$

which can be rewritten as:

$$\phi_j = \frac{1}{|\bar{\mathcal{E}}| - 1}\left[1 - b\left(\left(|\bar{\mathcal{E}}| - 1\right) p_j - \sum_{i\in\bar{\mathcal{E}}\setminus\{f\}} p_i\right)\right].$$

Substituting equation (7) into the above expression gives:

$$\phi_j = -b\left(p_j - p_f\right).$$

Therefore, if a firm in $\bar{\mathcal{E}}$ has zero market share then removing that firm from the market does not affect the market shares of the other firms. ∎

Lemma 4. *Suppose we have an arbitrary, but fixed vector of retail prices p, then under demand model $\mathcal{D}$, ϕ_f is a concave function of p_f, for $\phi_f > 0$.*

Proof: First note from lemma 3, the market-share of all firms j, ϕ_j, from $\mathcal{D}$ must be continuous functions of p_f, for each firm $f \in \mathcal{E}$.

Starting from any vector p such that $\phi_f > 0$ for all $f \in \mathcal{E}$, observe from the equation defining the retailers' market-share under demand model $\mathcal{D}$ (repeated below) that decreasing p_f will cause the demand of all firms, j in $\bar{\mathcal{E}}\setminus\{f\}$ to decrease:

$$\phi_j = \frac{1}{|\bar{\mathcal{E}}|}\left[1 - b \sum_{i\in\bar{\mathcal{E}}\setminus\{j\}} \left(p_j - p_i\right)\right].$$

The derivative with respect to p_f is:

$$\frac{\partial \phi_j}{\partial p_f} = \frac{b}{|\bar{\mathcal{E}}|},$$

which is positive, hence a decrease in p_f causes a decrease in ϕ_j for all $j \in \mathcal{E} \setminus \{f\}$.

Thus, decreasing p_f will lead to fewer firms in the market, since, under demand model $\mathcal{D}$, once x_j becomes negative, firm j will be removed from $\bar{\mathcal{E}}$.

Hence we can now examine the shape of ϕ_f as a function of p_f. First, for a fixed $\bar{\mathcal{E}}$ we will compute the derivative of ϕ_f with respect to p_f:

$$\frac{\partial \phi_f}{\partial p_f} = -b \frac{\left|\bar{\mathcal{E}}\right| - 1}{\left|\bar{\mathcal{E}}\right|}. \tag{9}$$

Note that this is negative, therefore, while the number of firms in the market is unchanged, ϕ_f is a decreasing function of p_f. However, from above we know that as p_f is decreased the number of firms in set $\bar{\mathcal{E}}$ monotonically decreases. From equation (9), we can see that the gradient increases (becomes less negative) as $\left|\bar{\mathcal{E}}\right|$ decreases, hence ϕ_f is a concave function over this range of p_f. ∎

LEMMA 5. *Consider the modified risk-adjusted profit maximization problem for firm f, $\tilde{\mathcal{P}}_f(p_{-f}, \epsilon)$, below (where ϵ is a vector with components ϵ_n):*

$$\begin{aligned}
\tilde{\Pi}_f(p_{-f}, \epsilon) = \max \; & \sum_{\omega \in \Omega} \rho \pi_f^\omega - \tfrac{\theta}{\alpha} \sum_{\omega \in \Omega} \rho \left[(1-\alpha) w_f^\omega + \alpha v_f^\omega\right] \\
\text{s.t. } & \pi_f^\omega = \eta_f + v_f^\omega - w_f^\omega && \left[\lambda_f^\omega\right] \; \forall \omega \in \Omega \\
& \pi_f^\omega \le \sum_{n \in \mathcal{N}} \left[X_n^\omega \sum_{i \in \mathcal{E}_n \cap \{f\}} \left(p_{ni} - c_n^\omega\right) \hat{\phi}_{ni} \right] + P_f^\omega && \left[\mu_f^\omega\right] \; \forall \omega \in \Omega \\
& \hat{\phi}_{nf} = \tfrac{1}{|\mathcal{E}_n|} \left[1 + b \sum_{i \in \mathcal{E}_n \setminus \{f\}} \left(p_{ni} - p_{nf}\right) \right] - \epsilon_n && \left[\nu_{nf}\right] \; \forall n \in \mathcal{N} \\
& w_f^\omega, v_f^\omega \ge 0 && \forall \omega \in \Omega \\
& \hat{\phi}_{nf} \ge 0 && \forall n \in \mathcal{N}.
\end{aligned}$$

Suppose we have some p_{-f} such that $\phi_{nf} > 0, \forall f \in \mathcal{E}_n, \forall n$ at the optimal solution to $\mathcal{P}_f(p_{-f})$. Then given a vector $\epsilon \ge 0$, we have $\tilde{\Pi}_f(p_{-f}, \epsilon) \le \Pi_f(p_{-f})$.

PROOF: We will prove this in two parts: first we will show that $\Pi_f(p_{-f}) = \tilde{\Pi}_f(p_{-f}, 0)$; and then that $\tilde{\Pi}_f(p_{-f}, \epsilon) \le \tilde{\Pi}_f(p_{-f}, 0)$, for all $\epsilon \ge 0$.

Note that there are three differences between $\mathcal{P}_f(p_{-f})$ and $\tilde{\mathcal{P}}_f(p_{-f}, \epsilon)$:

- the second constraint is now an inequality;
- ϵ appears in the third constraint; and
- a non-negativity constraint on $\hat{\phi}_{nf}$ has been added.

Since at the optimal solution to $\mathcal{P}_f\left(p_{-f}\right)$ we have (from the statement of the lemma) that $\hat{\phi}_{nf} > 0$, the addition of the non-negativity constraint does not change the optimal solution. Moreover, the third constraint in $\mathcal{P}_f\left(p_{-f}\right)$ and $\tilde{\mathcal{P}}_f\left(p_{-f}, 0\right)$ are identical since $\epsilon = 0$. The final concern is whether changing the second constraint to an inequality may lead to a different optimal solution. Through the KKT conditions[15] we will show that the second constraint is always binding at the optimal solution to $\tilde{\mathcal{P}}_f\left(p_{-f}, 0\right)$ and thus that solution must be the same as the optimal solution to $\mathcal{P}_f\left(p_{-f}\right)$.

We present the KKT conditions below:

$$\pi_f^\omega = \eta_f + v_f^\omega - w_f^\omega \quad \forall \omega \in \Omega \tag{10}$$

$$0 \leq \mu_f^\omega \perp \sum_{n\in\mathcal{N}} \left[X_n^\omega \sum_{i\in\mathcal{E}_n\cap\{f\}} \left(p_{ni} - c_n^\omega\right) \hat{\phi}_{ni} \right] + P_f^\omega - \pi_f^\omega \geq 0 \quad \forall \omega \in \Omega \tag{11}$$

$$\hat{\phi}_{nf} = \frac{1}{|\mathcal{E}_n|} \left[1 + b \sum_{i\in\mathcal{E}_n\setminus\{f\}} \left(p_{ni} - p_{nf}\right) \right] - \epsilon_n \quad \forall n \in \mathcal{N} \tag{12}$$

$$-\rho + \lambda_f^\omega + \mu_f^\omega = 0 \quad \forall \omega \in \Omega \tag{13}$$

$$-\sum_{\omega\in\Omega} \lambda_f^\omega = 0 \tag{14}$$

$$0 \leq v_f^\omega \perp \theta\rho - \lambda_f^\omega \geq 0 \quad \forall \omega \in \Omega \tag{15}$$

$$0 \leq w_f^\omega \perp \theta \frac{1-\alpha}{\alpha} \rho + \lambda_f^\omega \geq 0 \quad \forall \omega \in \Omega \tag{16}$$

$$-\sum_{\omega\in\Omega} X^\omega \hat{\phi}_{nf} \mu_f^\omega + b \frac{|\mathcal{E}_n| - 1}{|\mathcal{E}_n|} \nu_{nf} = 0 \quad \forall n \in \mathcal{N} \tag{17}$$

$$0 \leq \hat{\phi}_{nf} \perp \nu_{nf} - \sum_{\omega\in\Omega} \left[X_n^\omega \sum_{i\in\mathcal{E}_n\cap\{f\}} (p_{ni} - c_n^\omega) \mu_f^\omega \right] \geq 0 \quad \forall n \in \mathcal{N}. \tag{18}$$

First, consider right-hand inequalities of constraints (15) and (16). These can be rearranged to give the following upper and lower bounds on λ_f^ω:

$$-\theta \frac{1-\alpha}{\alpha} \rho \leq \lambda_f^\omega \leq \theta\rho.$$

From above and equation (13), we can therefore find bounds on μ_f^ω:

$$\rho\left(1-\theta\right) \leq \mu_f^\omega \leq \rho \left(1 + \theta \frac{1-\alpha}{\alpha}\right).$$

[15] Since the above mathematical program is convex, the KKT conditions are both necessary and sufficient for optimality.

Since $\theta < 1$, we can see that $\mu_f^\omega > 0$, $\forall \omega \in \Omega$. From the orthogonality condition (11) we therefore must have the second constraint binding at the optimal solution. Hence, since the optimal solutions are identical we must have

$$\Pi_f(p_{-f}) = \tilde{\Pi}_f(p_{-f}, 0).$$

Now let us turn our attention to the second part of the lemma. We wish to prove, for any $\epsilon \geq 0$, that $\tilde{\Pi}_f(p_{-f}, \epsilon) \leq \tilde{\Pi}_f(p_{-f}, 0)$

From the result above that $\mu_f^\omega > 0$ and equation (17) we can deduce the sign of ν_{nf}:

$$\nu_{nf} = \overbrace{\frac{|\mathcal{E}_n|}{b(|\mathcal{E}_n| - 1)}}^{\geq 0} \phi_{nf} \overbrace{\sum_{\omega \in \Omega} X^\omega \mu_f^\omega}^{\geq 0},$$

thus, so long as $\phi_{nf} \geq 0$, we have $\nu_{nf} \geq 0$. This means that the rate of change of the optimal value function with respect to any component of ϵ is negative. Moreover, this is true for any $\epsilon \geq 0$, hence we must have $\tilde{\Pi}(p_{-f}, \epsilon) \leq \tilde{\Pi}(p_{-f}, 0)$.

Combining this result with the one from the first part of the proof, we have:

$$\tilde{\Pi}_f(p_{-f}, \epsilon) \leq \Pi_f(p_{-f}), \quad \forall \epsilon \geq 0,$$

as required. ∎

Note that lemma 5 implies that any decrease in the (individual) nodal market-share (while keeping the prices fixed) will yield a lower profit compared to the optimal profit given the vector of current prices.

Lemma 6. *Given an arbitrary but fixed set of entrants $\mathcal{E}$ ($\mathcal{E}_n$ for each node $n \in \mathcal{N}$), any equilibrium, ψ, for $\hat{\mathcal{G}}_{\mathcal{E}}$ that has positive demand for all firms is also an equilibrium for $\mathcal{G}_{\mathcal{E}}$.*

Proof: First recall that ψ is an equilibrium for $\hat{\mathcal{G}}_{\mathcal{E}}$ if and only if no firm has incentive to deviate. We now consider the strategy set ψ under game $\mathcal{G}_{\mathcal{E}}$. For ψ to not be an equilibrium to $\mathcal{G}_{\mathcal{E}}$, this means at least one firm has incentive to deviate from ψ. Clearly this deviation (to strategy $\hat{\psi}$) must result in $\phi_{nf} = 0$ under game $\mathcal{G}_{\mathcal{E}}$ and $\phi_{nf} < 0$ under game $\hat{\mathcal{G}}_{\mathcal{E}}$ for at least one firm f and node n. We will analyse the following two cases:

1. either f, the deviating firm, causes negative demand (under $\hat{\mathcal{G}}_{\mathcal{E}}$) for itself (and possibly other firms),

2. or, firm f's deviation causes another firm to have negative demand (under $\hat{\mathcal{G}}_{\mathcal{E}}$).

We will show that both cases lead to a contradiction. Under case 1, $\hat{\psi}$ leads to a nodal market-share of 0 under $\mathcal{G}_{\mathcal{E}}$. Such a strategy gives a retail profit of 0 at node n for all scenarios. The total risk-adjusted profit in $\mathcal{G}_{\mathcal{E}}$ would be the same if the firm were to choose a price, p_{nf}, such that $\phi_{nf} = 0$ in $\hat{\mathcal{G}}_{\mathcal{E}}$; since this was a feasible choice for the firm in $\hat{\mathcal{G}}_{\mathcal{E}}$, this can not be a profitable deviation.

To show a contradiction for case 2, we must first understand the nature of a firm's market-share as a function of its retail price under $\mathcal{G}_{\mathcal{E}}$. Recall from lemma 4 that, in $\mathcal{G}_{\mathcal{E}}$, firm f's market-share at any node n as a function of its retail price is concave (so long as $\phi_{nf} > 0$), and observe from equation (2) that the market share in $\hat{\mathcal{D}}$ is a linear function of prices. Moreover, when $\phi_{ni} > 0, \forall i \in \mathcal{E}_n, \forall n \in \mathcal{N}$, the market share from both $\mathcal{G}_{\mathcal{E}}$ and $\hat{\mathcal{G}}_{\mathcal{E}}$ are identical. Therefore, for a given retail price, p_{nf}, the market share for firm f computed from $\mathcal{D}$ must be less than or equal to the market share from $\hat{\mathcal{D}}$.

Thus, suppose we compute the optimal solution to firm f's best response problem, assuming demand model $\mathcal{D}$ and this gave optimal prices, p^*_{nf}, and retail market-shares of ϕ^*_{nf}. Now set $\epsilon^*_n = \hat{\phi}_{nf}\left(p^*_{nf}\right) - \phi^*_{nf}$ (which must be non-negative due the concavity of ϕ_{nf}); it is clear that for $\tilde{\mathcal{P}}\left(p_{-f}, \epsilon^*\right)$, p^*_{nf} gives market-shares $\phi_{nf}\left(p^*_{nf}\right) - \epsilon^*_n = \phi^*_{nf}$ (thus the deviation profit is equal to $\tilde{\Pi}\left(p_{-f}, \epsilon^*\right)$). However, from lemma 5 we know that $\tilde{\Pi}\left(p_{-f}, \epsilon^*\right) \leq \Pi\left(p_{-f}\right)$. Thus there can be no profitable deviation.

The two possible deviation cases, considered above, have been shown to not give incentive for the deviation. Hence, since there are no other possible deviation strategies, ψ is an equilibrium for $\mathcal{G}_{\mathcal{E}}$. ∎

Lemma 7. *Any equilibrium, ψ, to the game $\mathcal{G}_{\mathcal{E}}$ which has positive demand for all firms is also an equilibrium to the game $\hat{\mathcal{G}}_{\mathcal{E}}$.*

Proof: First observe that in the game $\hat{\mathcal{G}}_{\mathcal{E}}$, each firm has a convex optimization problem. Moreover, note that demand models $\mathcal{D}$ and $\hat{\mathcal{D}}$ yield identical retail demands where $\phi > 0$. Hence any interior global optimal for a firm's profit maximization under demand model $\mathcal{D}$ must also be the global optimal under $\hat{\mathcal{D}}$. Since the equilibrium, ψ, for the game $\mathcal{G}_{\mathcal{E}}$ is such that $\phi_{nf} > 0$ for all firms and nodes, and each firm is at a global optimal strategy, such a point must be a equilibrium for the game $\hat{\mathcal{G}}_{\mathcal{E}}$. ∎

THEOREM 2. *If there exists no equilibrium for the game $\hat{\mathcal{G}}_{\mathcal{E}}$ with $\hat{\phi}_{nf} > 0$ for all firms f at all nodes n, then there exists no equilibrium for the game $\mathcal{G}_{\mathcal{E}}$ with $\phi_{nf} > 0$. Thus, if an equilibrium exists for $\mathcal{G}_{\mathcal{E}}$, then there must be at least one firm which receives no demand, and hence would have been better off not entering the retail market in the first stage (assuming a positive cost of entry).*

PROOF: From lemma 7 we know that any equilibrium for $\mathcal{G}_{\mathcal{E}}$ with $\phi_{nf} > 0$ is also an equilibrium for $\hat{\mathcal{G}}_{\mathcal{E}}$, thus if no such equilibrium exists for $\hat{\mathcal{G}}_{\mathcal{E}}$ then the same is true for $\mathcal{G}_{\mathcal{E}}$. Therefore, if an equilibrium for $\mathcal{G}_{\mathcal{E}}$ does exist, at least one firm will receive a market share of 0 at some node n. In this case, with a positive cost of entry, any such firm would be able to improve its total profit by exiting the retail market at node n. ∎

The above lemmas and theorem show that if we compute an equilibrium for $\hat{\mathcal{G}}_{\mathcal{E}}$ with positive demand for all firms and nodes then this is a valid equilibrium for $\mathcal{G}_{\mathcal{E}}$. Conversely, if we find that there is some firm with a negative market share at a node, then some firm would achieve a higher payoff by not entering the market in the first stage (meaning that this sub-game would not be on the equilibrium path).

Appendix

Conditional value at risk as a mean–risk measure

From Shapiro et al. (2009), the following function of the random variable Z gives the weighted mean deviation from any given quantile α

$$r_\alpha [z] = \min_\eta \{\varphi(\eta) := E[\max\{(1-\alpha)(\eta - z), \alpha(z-\eta)\}]\}.$$

To see that the optimal η does in fact correspond to the β-quantile, we first take left and right derivatives of the above function with respect to η:

$$\varphi'(\eta)_+ = \Pr[z \leq \eta] \times (1-\alpha) - \Pr[z > \eta] \times \alpha \geq 0,$$

$$\varphi'(\eta)_- = \Pr[z < \eta] \times (1-\alpha) - \Pr[z \geq \eta] \times \alpha \leq 0.$$

Observe that at the optimal η the left derivative must be non-increasing and the right derivative must be non-decreasing.[16] We can then rearrange the above inequalities to find:

$$\Pr[z < \eta] \leq \alpha \leq \Pr[z \leq \eta],$$

which confirms that the optimal η is the α-quantile.

Hence if $F(z)$ is the cumulative distribution function and $f(z)$ is the probability density function corresponding to the random variable z, then

$$\begin{aligned}
r_\alpha[z] &= \int_{F^{-1}(0)}^{F^{-1}(\alpha)} (1-\alpha)\left(F^{-1}(\alpha) - z\right) f(z)\,dZ + \int_{F^{-1}(\alpha)}^{F^{-1}(1)} \alpha\left(z - F^{-1}(\alpha)\right) f(z)\,dZ \\
&= \int_{F^{-1}(\alpha)}^{F^{-1}(1)} \alpha z f(z)\,dz - \int_{F^{-1}(0)}^{F^{-1}(\alpha)} (1-\alpha) z f(z)\,dz + (1-\alpha) F^{-1}(\alpha) \int_{F^{-1}(0)}^{F^{-1}(\alpha)} f(z)\,dz - \alpha F^{-1}(\alpha) \int_{F^{-1}(\alpha)}^{F^{-1}(1)} f(z)\,dz \\
&= \int_{F^{-1}(\alpha)}^{F^{-1}(1)} \alpha z f(Z)\,dz - \int_{F^{-1}(0)}^{F^{-1}(\alpha)} (1-\alpha) z f(Z)\,dz \\
&= (1-\alpha)\,\alpha\left(E_{z|z\geq F^{-1}(\alpha)}[z] - E_{z|z\leq F^{-1}(\alpha)}[z]\right).
\end{aligned}$$

Moreover, note that the expectation of z can be written as:

$$E[z] = \alpha E_{z|z\leq F^{-1}(\alpha)}[z] + (1-\alpha) E_{z|z\geq F^{-1}(\alpha)}[z].$$

Finally, we find that

$$\begin{aligned}
E[z] - \frac{1}{\alpha} r_\alpha[z] &= \alpha E_{z|z\leq F^{-1}(\alpha)}[z] + (1-\alpha) E_{z|z\geq F^{-1}(\alpha)}[z] - (1-\alpha)\left(E_{z|z\geq F^{-1}(\alpha)}[Z] - E_{Z|Z\leq F^{-1}(\alpha)}[z]\right) \\
&= \alpha E_{z|z\leq F^{-1}(\alpha)}[z] + (1-\alpha) E_{z|z\leq F^{-1}(\alpha)}[z] \\
&= E_{z|z\leq F^{-1}(\alpha)}[z] = -\mathrm{CVaR}_\alpha(z).
\end{aligned}$$

This can be incorporated into a mean-risk optimization problem with a parameter $\alpha \in [0,1]$ controlling the weightings on risk versus mean return.

$$\begin{aligned}
(1-\theta) E[z] - \theta \mathrm{CVaR}_\alpha(z) &= (1-\theta) E[z] + \theta\left(E[z] - \frac{1}{\alpha} r_\alpha[z]\right) \\
&= E[z] - \frac{\theta}{\alpha} r_\alpha[z].
\end{aligned}$$

References

B. Allaz and J-L. Vila. Cournot competition, forward markets and efficiency. *Journal of Economic Theory*, 59(1):1–16, 1993.

[16] Of course this assumes that φ is continuous in η, which can be easily verified.

P. Artzner, F. Delbaen, J.-M. Eber, and D. Heath. Coherent Measures of Risk. *Mathematical Finance*, pages 203–228, 1999.

Fridrik M. Baldursson and Nils-Henrik von der Fehr. Vertical Integration and Long-Term Contracts in Risky Markets. Memorandum 01/2007, Oslo University, Department of Economics, January 2007. URL `http://ideas.repec.org/p/hhs/osloec/2007_001.html`.

M.C. Caramanis. Investment Decisions and Long-term Planning under Electricity Spot Pricing. *IEEE Transactions on Power Apparatus and Systems*, 101(9):4640–4648, 1982.

D Carlton. Vertical Integration in Competitive Markets under Uncertainty. *The Journal of Industrial Economics*, 27(3):189–209, March 1979.

M. Carrion. Risk-Aversion Modeling: Medium-Term Electricity Retailer Decision-Making. In *Power and Energy Society General Meeting - Conversion and Delivery of Electrical Energy in the 21st Century*, pages 1–3. IEEE, August 2008.

M. Carrion, U. Gotzes, and R. Schultz. Risk Aversion for an Electricity Retailer with Second-order Stochastic Dominance Constraints. *Computational Management Science*, 6:233–250, 2009.

S. Choi and A. Ruszczynski. A Risk-Averse Newsvendor with Law Invariant Coherent Measures of Risk. *Operations Research Letters*, 36:77–82, 2008.

A. Downward, D. Young, and G. Zakeri. Swapping Generators' Assets: Market Salvation or Wishful Thinking. *The Energy Journal*, 32(2), 2011.

Electricity Technical Advisory Group. Improving Electricity Market Performance. A preliminary report to the Ministerial Review of Electricity Market Performance, 2009.

M. C. Ferris and T. S. Munson. Complementarity Problems in GAMS and the PATH Solver. *Journal of Economic Dynamics and Control*, 24(2):165–188, 2000.

H. Markowitz. *Portfolio Selection.* John Wiley & Sons, 1959.

Karsten Neuhoff and Laurens De Vries. Insufficient Incentives for Investment in Electricity Generations. *Utilities Policy*, 12:253–267, 2004.

Andrew Powell. Trading Forward in an Imperfect Market: The Case of Electricity in Britain. *The Economic Journal*, 103:444–453, March 1993.

A. Shapiro, D. Dentcheva, and A. Ruszczynski. *Lectures of Stochastic Programming: Modeling and Theory.* MPS-SIAM, 2009.

X. Vives. *Oligopoly Pricing: Old Ideas and New Tools.* MIT Press, 1999.